A BASIC INVESTIGATION

OF

PERSPECTIVE MAP PROJECTIONS

A THESIS

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1. INTRODUCTION

1. Map Projections

The subject of map projections has developed from its inception sometime early in the history of mankind, consisting of perhaps a rough sketch drawn in the sand, to a specialized branch of applied mathematics which it is today. Due both to this long development and to the extent of the subject, it is possible to encounter numerous definitions of a map projection. The very name of the subject, map projections, is deceiving, for most map projections are not projections in the true mathematical sense of the word. In the true mathematical sense, the term projection has the connotation perspective projection. Perspective projection may be simply described as the determining of the position on a plane of points in space as they would appear on the plane from some particular fixed view point. Applying the above definition to map projection would indicate that a map projection is the representation of the surface of the earth, or an accepted datum surface approximating the surface of the earth, on a plane as it would appear on the plane from some particular fixed view point. However, the above definition can certainly not be used to totally describe map projection in the sense in which the term is commonly used. The term projection, as applied to map projection, includes numerous geometrical, mathematical, and semi-geometrical transformations of a datum surface onto a plane. Only a very few of these projections are truly perspective projections in a mathematical sense.

As was stated earlier, a map projection has been defined in many ways. One method of describing a map projection is to describe the transformation of the meridians and parallels from the surface of the earth, or a datum surface approximating the surface of the earth, to a mapping plane. The author will define a map projection as a systematic arrangement of intersecting lines on a plane that represent, and have a one-to-one correspondence to, the meridians and parallels of the datum surface. These lines, drawn to represent the meridians and parallels, are drawn according to some consistent principle in order to fulfill certain required conditions. Each set of new conditions produces a different map projection, and hence there potentially exists an unlimited number of map projections. Common conditions used in map projections are such that distances, areas, or angles measured on the datum surface will be equal to those measured on the mapping plane. In addition to the more common conditions stated above, a myriad of other conditions have been used to produce map projections. The conditions by which each particular map projection is designed are naturally based on the intended purpose of the map.

1.2 Perspective Map Projection

Based on the discussion in Section 1.1, it is apparent that the perspective map projection is only one of the many different types of map projections. It has characteristics, both desirable and undesirable, which dictate its use or non-use for particular purposes. A perspective map projection may be ideally vizualized as a

photograph taken of a portion of the datum surface. A photograph taken by an aircraft, a rocket, or a space vehicle of the earth, moon, or other heavenly body is actually a type of perspective projection. This certainly is one of the interesting aspects of the study of perspective projections.

Perspective projections are based on simple geometric principles. A straight line, which will be called the projection axis of the projection, extends from a point on the datum surface, through a point in space named the projection center, to the projection plane. This projection axis is perpendicular both to the datum surface and the projection plane. The intersection of the projection axis and the projection plane locates a point denoted as the image center. Likewise, all other points on the datum surface are located on the projection plane by extending straight lines from the particular point, through the projection center, onto the projection plane. In Figure 1, C represents the projection center and C', the image center. P is an arbitrary point on the datum surface and P' is the projection of this point on the projection plane. Principles of elementary geometry prove that a parallel displacement of the projection plane along the projection axis changes only the scale of the projection. However, a change in the location of the projection center will change the form of the projection. The location of the projection center is the factor which determines the characteristics of a particular perspective projection.

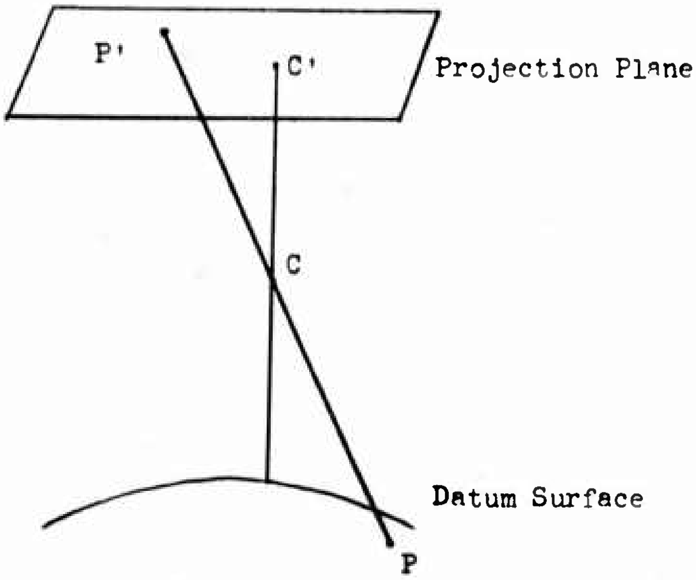

Figure 1

One means of classifying perspective projections is according to the orientation of the projection axis. If the projection axis is coincident with the axis of rotation of the datum surface, the projection is referred to as a normal or polar projection. If the projection axis lies in the equatorial plane of the datum surface, the projection is referred to as a transverse or equatorial projection. If the projection falls into neither of the above two special categories, then it is referred to as an oblique projection. It should be kept in mind that the normal and transverse projections are merely special cases of the oblique projection and equations derived for oblique projections are applicable to all special cases.

1.3 The Investigation Of Perspective Map Projections

The purpose of this thesis is to investigate the perspective projection in a very basic manner. The primary goal is to develop general mapping equations for the perspective projection suitable both for a spherical and an ellipsoidal datum surface. The term, mapping equations, refers to equations which transform the coordinates of a point from a latitude-longitude-elevation three dimensional type coordinate system of the datum surface to a X-Y plane coordinate system of the projection plane. A discussion of the distortions inherent in perspective projections is included to illustrate a method of using distortions to describe a projection. It will be seen that a combination coordinate-distortion type table or grid is an effective aid in vizualizing a projection.

The investigation is divided into five primary parts. In order to provide general background material and also provide equations on which to base the validity of equations derived later, Chapter 2 will briefly discuss and derive in a conventional manner the mapping equations for the three most common types of perspective projections. These three common projections are the Gnomonic Projection, the Stereographic Projection, and the Orthographic Projection. Chapter 3 discusses a descriptive geometry approach to perspective map projections as presented by Erwin Schmid [12]. It is from this publication that the author first became interested in the subject matter of this thesis. Chapter 4 is concerned with the actual development of general mapping equations for the

perspective projection. These equations are developed utilizing a photogrammetric approach, an approach which to the author's knowledge is novel to the study of map projections. Equations are first derived using the sphere as a datum surface in order both that the validity of the equations may be checked by comparison with those equations derived earlier and to facilitate for the reader the understanding of later derivations. In an analogous manner equations are then derived using the ellipsoid as a datum surface. It is shown that the mapping equations derived for an ellipsoidal datum surface are truly General Perspective Projection Mapping Equations, suitable for both spherical and ellipsoidal datum surfaces.

Chapter 5 briefly discusses an empirical method to determine the distortions inherent in the perspective projection of a latitude-longitude grid of a datum surface onto a projection plane. Chapter 6 illustrates examples of six types of perspective projections by means of tables or grids containing both X-Y plane coordinates and distortions. The coordinates and distortions are calculated for the intersection points of latitude and longitude lines spaced ten degrees apart. It is believed that the examples included in Chapter 6 both aid in the vizualization of perspective projections and at least partially validate the equations derived in Chapter 4. The computer program used to produce the examples of Chapter 6 and a brief explanation of the computer program are included in the Appendix.

2. A CONVENTIONAL APPROACH TO PERSPECTIVE MAP PROJECTIONS

Three of the more common perspective map projections are the Gnomonic Projection, the Stereographic Projection, and the Orthographic Projection. The mapping equations for these three projections are briefly derived in this chapter for two main reasons. First, the derivations will illustrate what the author chooses to call a conventional approach to the derivation of perspective mapping equations, similar to the approach used in [4]. Second, the equations derived in this chapter will be a basis of comparison for those equations derived later.

2.1 The Gnomonic Projection

The Gnomonic Projection is developed by placing a projection center at the center of a sphere and projecting this sphere onto a projection plane tangent to it. Figure 2 represents a sphere with a radius R and a center at C, which in the Gnomonic Projection is also the projection center. The projection plane AB is made tangent to the sphere at O' and this point is the image center of the projection. O' is located at a latitude ϕ_c on the sphere. P is an arbitrary point located on the sphere at a latitude ϕ and at a longitude difference of λ from O'. P' is the projection of P on the projection plane.

Figure 2 illustrates the following:

$CO' = CP = R$

Angle $O'NP = \lambda$

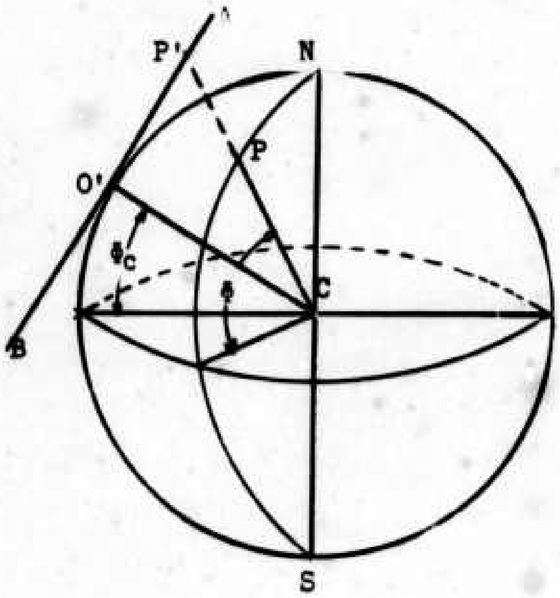

Figure 2

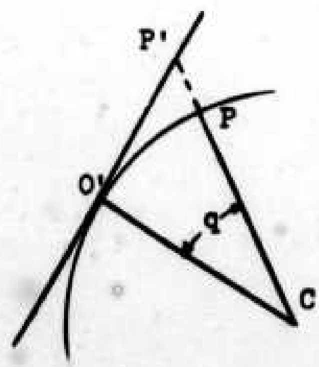

Fig.re 3

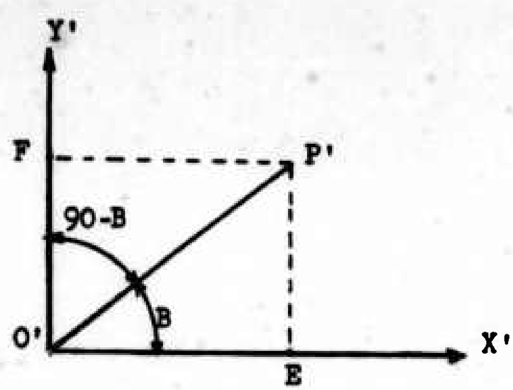

Figure 4

Angle NO'P = 90 - B

Arc NO' = $90 - \phi_c$

Arc NP = $90 - \phi$

Figure 3 is drawn in the plane of the triangle CO'P and illustrates the projection of the arc O'P onto the projection plane AB. From Figure 3, it is evident that

(1) $O'P' = R \tan q$

To reduce equation (1) to rectangular coordinates on the projection plane, the coordinate system illustrated in Figure 4 is developed. In this coordinate system, O'Y' represents the projection of the meridian through O' and O'X' represents the projection of the great circle through O' which is perpendicular to the meridian through O'.

From Figure 4, the following is derived:

(2) $X' = FP' = O'P' \cos B = R \tan q \cos B = \dfrac{R \sin q \cos B}{\cos q}$

(3) $Y' = EP' = O'P' \sin B = R \tan q \sin B = \dfrac{R \sin q \sin B}{\cos q}$

Utilizing the properties of the spherical triangle in Figure 1, the following may be derived by the Law of Sines:

$$\frac{\sin \lambda}{\sin q} = \frac{\sin (90 - B)}{\sin (90 - \phi)} = \frac{\cos B}{\cos \phi}$$

and therefore

(4) $\sin q \cos B = \sin \lambda \cos \phi$

By the Law of Cosines, the following is derived:

(5) $\sin q \sin B = \cos \phi_c \sin \phi - \sin \phi_c \cos \phi \cos \lambda$

(6) $\cos q = \sin \phi_c \sin \phi + \cos \phi_c \cos \phi \cos \lambda$

By substituting equations (4), (5), and (6) into equations (2) and (3), the mapping equations for the Gnomonic Projection are obtained:

$$X' = \frac{R \cos \phi \sin \lambda}{\sin \phi_c \sin \phi + \cos \phi_c \cos \phi \cos \lambda} \tag{7}$$

$$Y' = \frac{R (\cos \phi_c \sin \phi - \sin \phi_c \cos \phi \cos \lambda)}{\sin \phi_c \sin \phi + \cos \phi_c \cos \phi \cos \lambda} \tag{8}$$

2.2 The Stereographic Projection

The Stereographic Projection is developed by placing a projection center at a point on the sphere which is diametrically opposite the point at which the projection plane is tangent to the sphere. The mapping equations for the Stereographic Projection are developed in an analogous manner to the development of the mapping equations for the Gnomonic Projection. Figure 5 represents a sphere with a radius R and a center at O. The projection center is located at C and the projection plane AB is tangent to the sphere at O'. O' is also the image center of the projection. P is an arbitrary point located on the sphere at a latitude ϕ and at a longitude difference of λ from O'. P' is the projection of P on the projection plane.

Figure 5 illustrates the following:

$OC = OO' = R$

Angle $O'NP = \lambda$

Angle $NO'P = 90 - B$

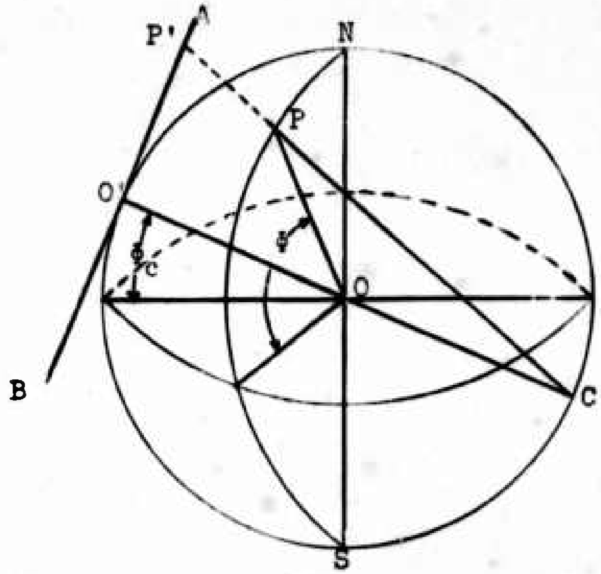

Figure 5

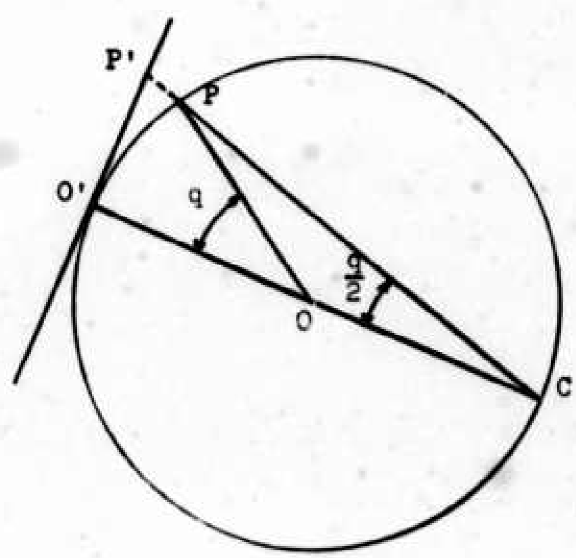

Figure 6

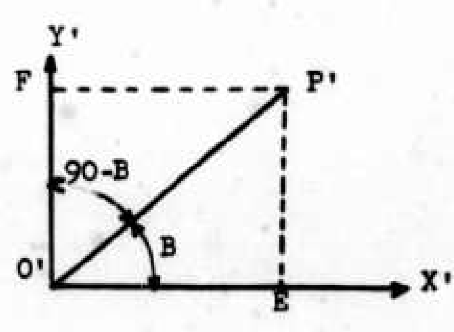

Figure 7

$$\text{Arc } NO' = 90 - \phi_c$$

$$\text{Arc } NP = 90 - \phi$$

Figure 6 is drawn in the plane of the triangle CO'P and illustrates the projection of the arc O'P onto the projection plane. It is evident from Figure 6 that

$$O'P' = 2\ R \tan \frac{q}{2} \tag{9}$$

To reduce equation (9) to rectangular coordinates on the projection plane, the coordinate system illustrated in Figure 7 is developed. In this coordinate system, O'Y' represents the projection of the meridian through O' and O'X' represents the projection of the great circle through O' which is perpendicular to the meridian through O'.

From Figure 7, the following is derived:

$$\begin{aligned} X' &= F'P' = O'P' \cos B = 2\ R \tan \frac{q}{2} \cos B \\ &= \frac{2\ R \sin q \cos B}{1 + \cos q} \end{aligned} \tag{10}$$

$$\begin{aligned} Y' &= E'P' = O'P' \sin B = 2\ R \tan \frac{q}{2} \sin B \\ &= \frac{2\ R \sin q \sin B}{1 + \cos q} \end{aligned} \tag{11}$$

Utilizing the spherical triangle formed in Figure 5, equations (4), (5), and (6) can again be derived. By substituting equations (4), (5), and (6) into equations (10) and (11), the mapping equations for the Stereographic Projection are obtained:

$$X' = \frac{2\ R \cos \phi \sin \lambda}{1 + \sin \phi_c \sin \phi + \cos \phi_c \cos \phi \cos \lambda} \tag{12}$$

$$Y' = \frac{2R(\cos\phi_c \sin\phi - \sin\phi_c \cos\phi \cos\lambda)}{1 + \sin\phi_c \sin\phi + \cos\phi_c \cos\phi \cos\lambda} \tag{13}$$

2.3 The Orthographic Projection

The Orthographic Projection is developed by placing a projection center at infinity and by projecting the datum sphere onto a projection plane perpendicular to the projection rays. All projection rays are parallel to each other and perpendicular to the projection plane. The mapping equations for the Orthographic Projection can be developed in analogous manner to the development of the mapping equations in Section 2.1 and Section 2.2. Figure 8 represents a sphere with a radius R and a center at O. The projection center is located at infinity in the direction indicated by the figure. The projection plane is perpendicular to the projecting rays and tangent to the sphere at O', with O' also being the image center of the projection. P is an arbitrary point located on the sphere at a latitude ϕ and at a longitude difference of λ from O'. P' is the projection of P on the projection plane.

Figure 8 illustrates the following:

$OO' = OP = R$

Angle $O'NP = \lambda$

Angle $NO'P = 90 - B$

Arc $NO' = 90 - \phi_c$

Arc $NP = 90 - \phi$

Figure 9 is drawn in the plane of triangle OO'P and illustrates the projection of the arc O'P onto the projection plane.

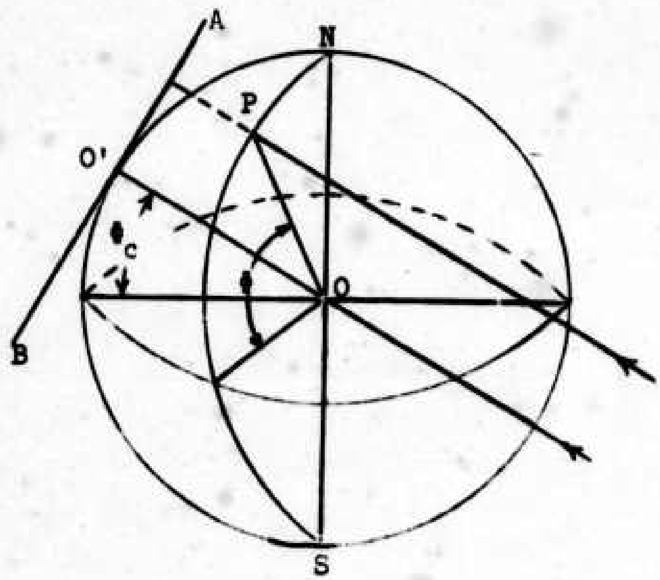

Figure 8

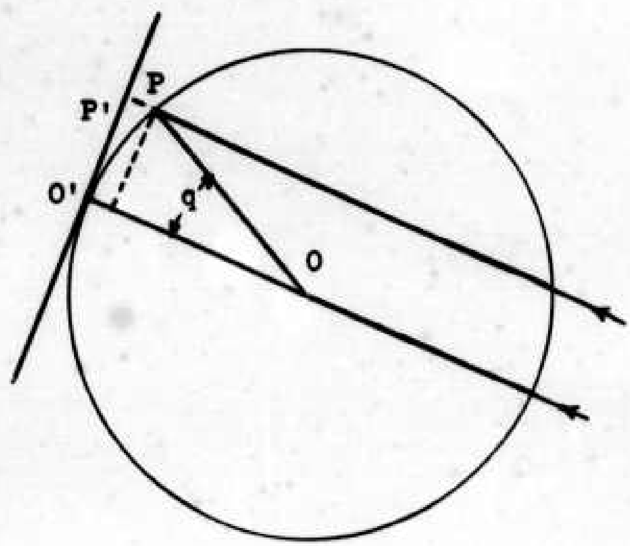

Figure 9

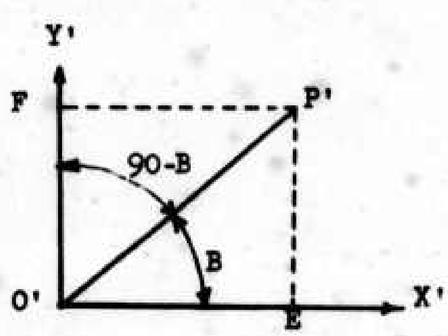

Figure 10

It is evident from Figure 9 that

(14) $$O'P' = R \sin q$$

To reduce equation (14) to rectangular coordinates on the projection plane, the coordinate system illustrated in Figure 10 is developed. In this coordinate system, O'Y' represents the projection of the meridian through O' and O'X' represents the projection of the great circle through O' which is perpendicular to the meridian through O'.

From Figure 10, the following is derived:

(15) $$X' = FP' = O'P' \cos B = R \sin q \cos B$$

(16) $$Y' = EP' = O'P' \sin B = R \sin q \sin B$$

Utilizing the spherical triangle formed in Figure 8, equations (4) and (5) can again be derived. By substituting equations (4) and (5) into equations (15) and (16), the mapping equations for the Orthographic Projection are obtained:

(17) $$X' = R \cos \phi \sin \lambda$$

(18) $$Y' = R (\cos \phi_c \sin \phi - \sin \phi_c \cos \phi \cos \lambda)$$

3. A DESCRIPTIVE GEOMETRY APPROACH TO PERSPECTIVE MAP PROJECTIONS

The three perspective map projections just discussed, the Gnomonic, the Stereographic, and the Orthographic, were developed by placing the projection center at a particular location in reference to the datum sphere. In this chapter the projection center will be placed at an arbitrary distance above the surface of the datum sphere and perspective mapping equations will be developed for this spherical datum surface. Since the projection center will be located at an arbitrary distance above the datum surface, it is now possible to better vizualize the physical significance of the perspective projection by comparing it to an aerial photograph. As was stated earlier, a photograph is actually a type of perspective projection. Thus, the perspective mapping equations derived in this chapter would enable the plotting of grid lines on an ideal vertical photograph taken at a particular distance above a datum sphere. Naturally, this can not be accomplished with an actual aerial photograph of the earth due to the fact that the earth is not a sphere nor a simple geometric figure of any type. Also, due to numerous distortions inherent to photography, aerial photographs are not ideal vertical projections. However, the comparison between perspective projections and vertical aerial photographs is still useful in visualizing the derivations included in this and the following chapter.

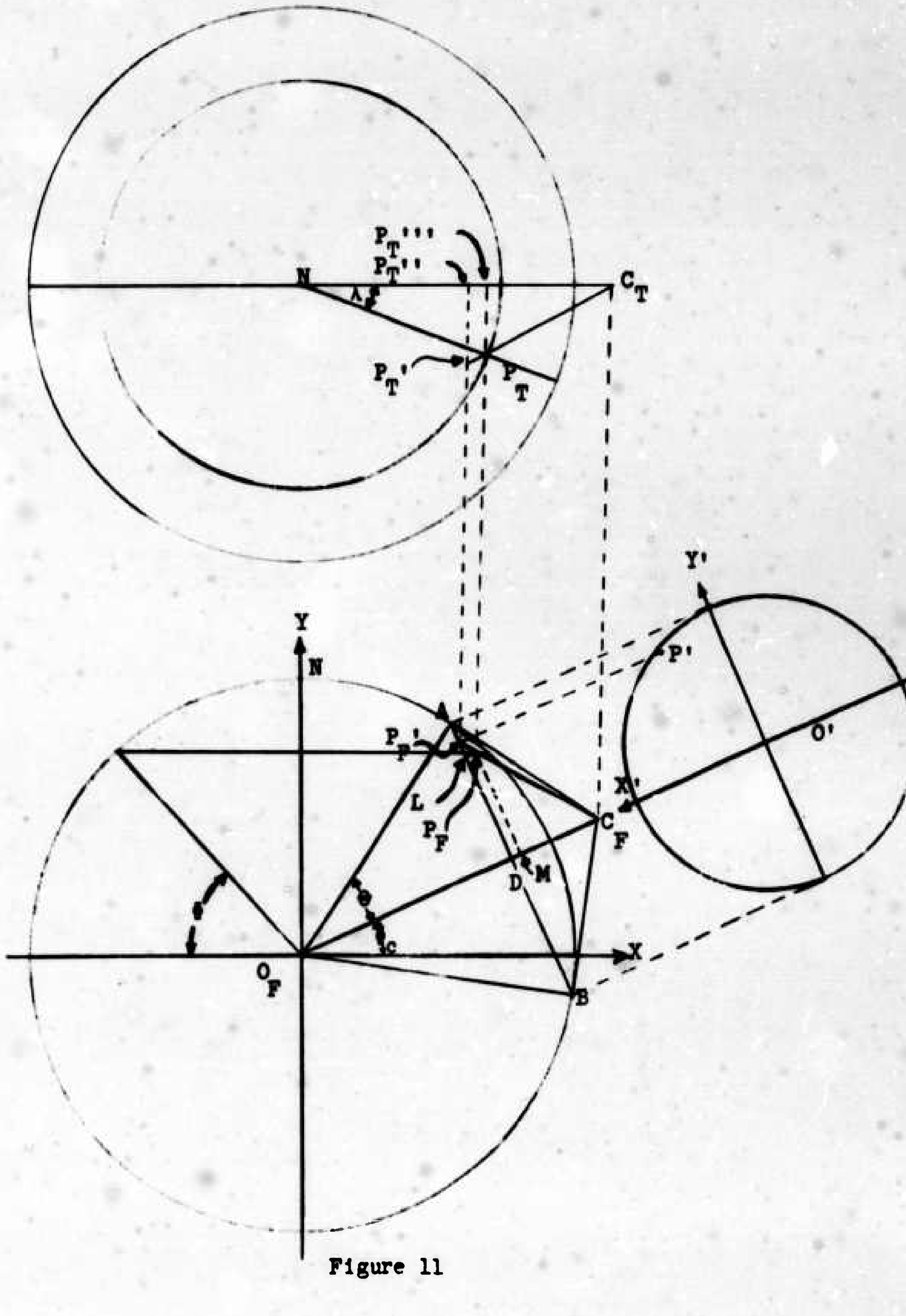

Figure 11

Figure 11 illustrates the descriptive geometry approach to perspective map projections presented in [12]. In this figure the lower circle represents a transverse or front view of the datum sphere, with the plane of the X axis defining the plane of the equator and the Y axis defining the polar axis. The upper circle in Figure 11 represents the polar or top view of the datum sphere, with N denoting the position of the pole. The small circle on the right side of the figure represents the perspective view of the datum sphere that would be obtained by placing a projection center at C_F. In this chapter mapping equations will be derived which allow the transformation of coordinates of an arbitrary point from a three dimensional latitude-longitude system, as illustrated in the top and front views of Figure 11, to a plane X'-Y' system, as illustrated on the projection plane in the perspective view of Figure 11.

In Figure 11, C_F represents a projection center located at a latitude ϕ_c, at a longitude λ_c, and at a distance h above a datum sphere of radius R. A cone of projection rays emanating from C_F is tangent to the sphere at A and B and creates right angles $O_F A C_F$ and $O_F B C_F$. The complete intersection of the projection cone and the sphere is a circle whose trace AB is shown in the front view. This circular intersection as seen from the projection center C_F is illustrated in the perspective view. This circular area can be vizualized as that portion of the datum sphere which could be seen if an observer were located at C_F. In this derivation the AB

plane will be used as the projection plane and all points on the visible portion of the sphere will be projected onto this plane.

In the front view of Figure 11, let P_F be an arbitrary point located at a latitude ϕ and at a longitude λ on the portion of the sphere visible from C_F. By definition, λ equals the difference in longitude between the arbitrary point P and the projection center C. P_T represents the position of the point P in the top view. P_F is projected along the projection ray C_FP_F until the ray intersects the projection plane [illegible] at P_F'. P_F' is then the projected image of P_F on the projection plane. P_T' represents the position of this point in the top view. It is seen that the distance DP_F' is then the Y' coordinate of the point P' on the projection plane. In the top view, the distance $P_T''P_T'$ is the X' coordinate of the point P' on the projection plane and this distance, when transferred to the perspective view, locates the point P'. P', with an X' value of $P_T''P_T'$ and a Y' value of DP_F', is thus the projection of point P on the projection plane in the perspective view.

The point P has now been transferred to the projection plane, and it only remains necessary to resolve the ordinate and abscissa values of the point into terms involving the original given parameters. This is accomplished in the following paragraphs.

The angle C_FAO_F is designated θ and it is evident that

$$\cos\theta = \frac{AO_F}{O_FC_F} = \frac{R}{R+h} \tag{19}$$

and also that

(20) $$O_F D = R \cos\theta$$

The equation of the line AB may now be written as

(21) $$X \cos \phi_c + Y \sin \phi_c - R \cos \theta = 0$$

and the equation of the line $O_F C_F$ may be written as

(22) $$X \sin \phi_c - Y \cos \phi_c = 0$$

Next, the two perpendicular distances LP_F and $P_F M$ from point P_F to lines AB and $O_F C_F$ respectively must be determined. This is done by using well known geometric formulas for finding the perpendicular distance from a point to a line. The resulting equations are

(23) $$LP_F = R (\cos \phi_c \cos \phi \cos \lambda + \sin \phi_c \sin \phi - \cos \theta)$$

(24) $$MP_F = -R (\sin \phi_c \cos \phi \cos \lambda - \cos \phi_c \sin \phi)$$

Considering the X'-Y' coordinate system of the projection plane in the perspective view, it was shown previously that the ordinate of P' in this coordinate system was equal to DP_F'. Utilizing similar triangles in the front view the following is obtained:

$$DP_F' = DC_F \frac{MP_F}{KC_F} = \frac{DC_F \cdot MP_F}{DC_F - LP_F}$$

and also from the front view the following is derived:

$$DC_F = AD \tan \theta = R \sin \theta \tan \theta$$

A combination of the above equations yields the Y' cordinate of P' in the X'-Y' coordinate system of the projection plane.

$$Y' = DP_F'$$

$$Y' = \frac{(R \sin \theta \tan \theta)\, R\, (\cos \phi_c \sin \phi - \sin \phi_c \cos \phi \cos \lambda)}{R \sin \theta \tan \theta - R(\cos \phi_c \cos \phi \cos \lambda + \sin \phi_c \sin \phi - \cos \theta)}$$

$$(25) \qquad Y' = \frac{R \sin^2 \theta\, (\cos \phi_c \sin \phi - \sin \phi_c \cos \phi \cos \lambda)}{1 - \cos \theta\, (\cos \phi_c \cos \phi \cos \lambda + \sin \phi_c \sin \phi)}$$

A X' coordinate corresponding to the Y' coordinate calculated by equation (25) may now be derived. Considering Figure 11 again, the following proportions are obtained:

$$\frac{P_T''P_T'}{P_T'''P_T} = \frac{P_T'C_T}{P_TC_T} = \frac{P_F'C_F}{P_FC_F} = \frac{DP_F'}{MP_F}$$

It is also evident from Figure 11 that

$$P_T'''P_T = R \cos \phi \sin \lambda$$

$$DP_F' = Y'$$

The abscissa X' of the point P' on the projection plane was previously shown to be equal to $P_T''P_T'$ and thus

$$X' = P_T''P_T' = \frac{P_T'''P_T \cdot DP_F'}{MP_F}$$

$$X' = \frac{R \cos \phi \sin \lambda \cdot Y'}{-R\,(\sin \phi_c \cos \phi \cos \lambda + \sin \phi_c \sin \phi)}$$

$$(26) \qquad X' = \frac{R \sin^2 \theta \cos \phi \sin \lambda}{1 - \cos \theta\, (\cos \phi_c \cos \phi \cos \lambda + \sin \phi_c \sin \phi)}$$

Summarizing, equations (25) and (26) are perspective

projection mapping equations which enable an arbitrary point to be transferred from a three dimensional latitude-longitude coordinate system of a datum sphere to a plane X'-Y' coordinate system of a projection plane.

4. A PHOTOGRAMMETRIC APPROACH TO PERSPECTIVE MAP PROJECTIONS

Equations (25) and (26), developed by a descriptive geometry method in Chapter 3, are mapping equations suitable for a spherical datum surface. It is not apparent to the author how these equations can be modified to make them applicable to an ellipsoidal datum surface. It is also not immediately apparent how these equations can be used for all possible locations of the projection center. In this chapter mapping equations will be derived that can be utilized regardless of the location of the projection center and that can be used both on spherical and ellipsoidal datum surfaces.

Since a perspective projection is directly comparable to a photograph, the author chose to derive the desired mapping equations through a photogrammetric approach. Section 4.1 deals with the derivation of equations based on a spherical datum surface, while Section 4.2 deals with similar equations based on an ellipsoidal datum surface. Since a sphere is only a special case of an ellipsoid, mapping equations derived for the sphere are only special cases of those derived for the ellipsoid. The author considers the spherical case worthy of separate discussion based on the belief that the sphere is a logical intermediate step before progressing to the more complicated ellipsoid. The mapping equations derived for the sphere may also be compared with previously derived equations as a check on the validity of the general method being utilized in this chapter.

4.1 A Spherical Datum Surface

4.11 The Mapping Equations

Figure 12 represents a spherical datum surface with a radius R and a conventional three dimensional X-Y-Z coordinate system. Point C depicts the projection center and is located at a latitude ϕ_c, at a longitude λ_c, and at a height above the datum sphere of h. Point P represents an arbitrary point on the datum surface and is located at a latitude ϕ and at a longitude λ. In the following derivation λ will equal the difference in longitude between the point P and the projection center C.

The space rectangular coordinates of point C are obtained from Figure 12.

(27) $$Y_c = (R + h) \cos \phi_c$$

(28) $$X_c = 0$$

(29) $$Z_c = (R + h) \sin \phi_c$$

Likewise, the coordinates of point P are obtained from Figure 12.

(30) $$Y_p = R \cos \phi \cos \lambda$$

(31) $$X_p = R \cos \phi \sin \lambda$$

(32) $$Z_p = R \sin \phi$$

As an aid in vizualizing the remainder of the derivation, point C may be compared to a camera lens and CO, the projection axis, is comparable to a camera axis. The projection plane AB, or mapping plane, may be vizualized as a photographic plate which is perpendicular to the camera axis and located at a distance f

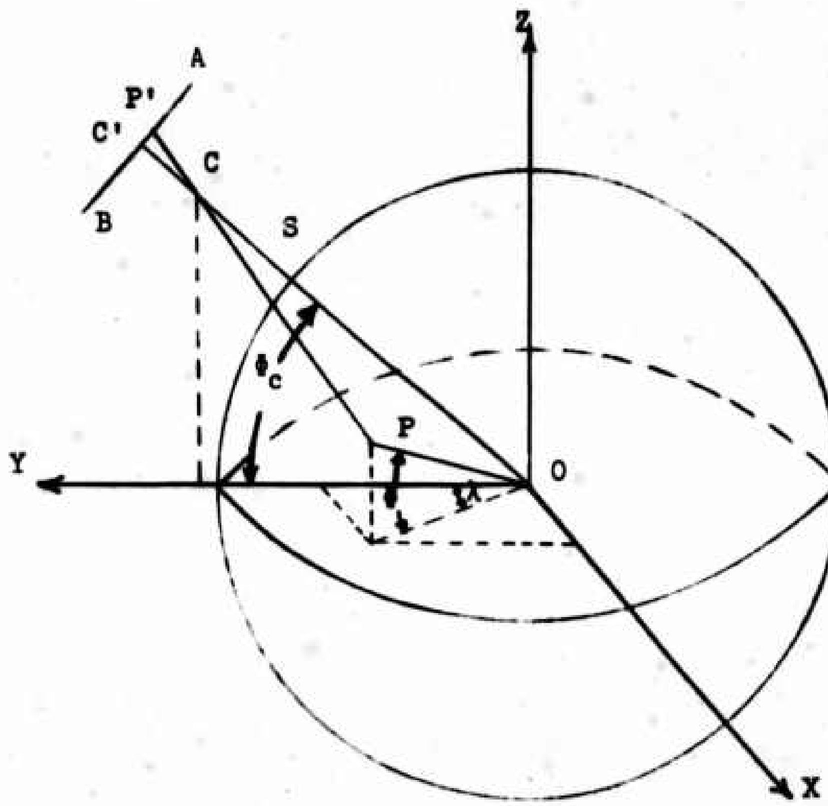

Figure 12

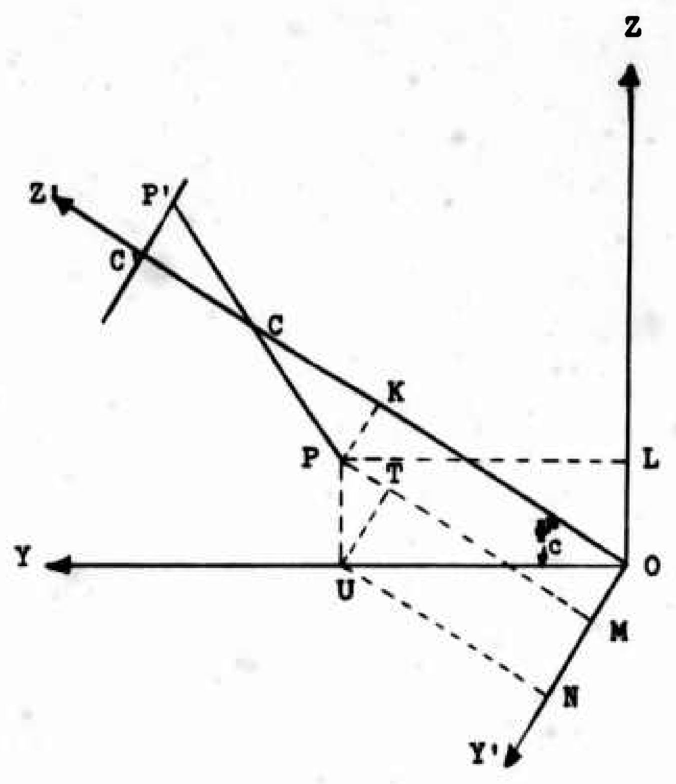

Figure 13

behind the lens. It is seen that f is comparable to the focal length of a camera and, thus, the scale of the projection is dependent on f. Actually, the ratio of f to h determines the scale of the projection; and it is apparent that a full scale projection will result when f is equal to h. It will be necessary later in the derivation to have firm definitions of both h and f as regards to sign convention. The signs of h and f are defined as positive when they are above, or to the outside of, that portion of the sphere to be projected and negative if to the inside of the sphere.

Figure 12 illustrates the following:

$CS = h$

$CC' = f$

$OS = OP = R$

The next step in the derivation is to rotate the original Z axis in the Z-Y plane until the Z axis coincides with the projection axis OC, as is illustrated in Figure 13. This rotation in the Z-Y plane creates a new X'-Y'-Z' coordinate system. It is evident that this type of rotation will not alter the X coordinate of points from their values as calculated in the original coordinate system.

Figure 13 illustrates the following:

$PL = Y_p$

$PU = Z_p$

$PK = Y_p'$

$PM = Z_p'$

$NO = Y_p \sin \phi_c$

$UT = Z_p \cos \phi_c$

$UN = Y_p \cos \phi_c$

$PT = Z_p \sin \phi_c$

Considering Figure 13, the space rectangular coordinates of point C in the new X'-Y'-Z' coordinate system are obtained.

(33) $$X_c' = 0$$

(34) $$Y_c' = 0$$

(35) $$Z_c' = R + h$$

Likewise, the new coordinates of point P are obtained.

(36) $$Y_p' = Y_p \sin \phi_c - Z_p \cos \phi_c$$

(37) $$X_p' = X_p$$

(38) $$Z_p' = Y_p \cos \phi_c + Z_p \sin \phi_c$$

Substituting equations (30), (31), and (32) into equations (36), (37), and (38), the X'-Y'-Z' coordinates of point P in terms of the original parameters are obtained.

(39) $$Y_p' = R \sin \phi_c \cos \phi \cos \lambda - R \cos \phi_c \sin \phi$$

(40) $$X_p' = R \cos \phi \sin \lambda$$

(41) $$Z_p' = R \cos \phi_c \cos \phi \cos \lambda + R \sin \phi_c \sin \phi$$

The equation of the projection plane AB which is perpendicular to the projection axis and hence now perpendicular to the Z' axis is clearly

(42) $$Z' = R + h + f$$

The equation of the projection ray from P that passes through C and intersects the projection plane at P' can be written as

(43) $$\frac{X' - X_p'}{X_c' - X_p'} = \frac{Y' - Y_p'}{Y_c' - Y_p'} = \frac{Z' - Z_p'}{Z_c' - Z_p'}$$

Equation (43) is the equation of the line in space along which P is projected onto the projection plane. The intersection of this line and the projection plane, as given by equation (42), is the location of P', the projection of P on the projection plane. Therefore, the simultaneous solution of equations (42) and (43) will yield the desired X'-Y' coordinates of the projected point.

The equation for X' is derived in the following manner:

From equation (43)

$$\frac{X' - X_p'}{X_c' - X_p'} = \frac{Z' - Z_p'}{Z_c' - Z_p'}$$

(44) $$X' = \frac{(Z' - Z_p')(X_c' - X_p')}{(Z_c' - Z_p')} + X_p'$$

Substituting equation (33) into equation (44)

(45) $$X' = \frac{X_p' \; (Z_c' - Z')}{Z_c' - Z_p'}$$

Substituting equations (35), (40), (41) and (42) into equation (45)

$$X' = \frac{- f \, R \cos \phi \sin \lambda}{R \, (1 - \cos \phi_c \cos \phi \cos \lambda - \sin \phi_c \sin \phi) + h}$$

Remembering that a photogrammetric approach was utilized in this derivation, it will be realized that the X' coordinate has suffered a reversal of sign during the process, presenting a reversed image of the original datum surface. In order to compensate for this situation the sign of the above equation must be reversed, and therefore

(46) $$X' = \frac{f \, R \cos \phi \sin \lambda}{R \, (1 - \cos \phi_c \cos \phi \cos \lambda - \sin \phi_c \sin \phi) + h}$$

The equation for Y' is derived in a manner analogous to the derivation of the equation for X' as follows:

From equation (43)

$$\frac{Y' - Y_p'}{Y_c' - Y_p'} = \frac{Z' - Z_p'}{Z_c' - Z_p'}$$

(47) $$Y' = \frac{(Z' - Z_p')(Y_c' - Y_p')}{(Z_c' - Z_p')} + Y_p'$$

Substituting equation (34) into equation (47)

(48) $$Y' = \frac{Y_p' \; (Z_c' - Z')}{Z_c' - Z_p'}$$

Substituting equations (35), (39), (41), and (42) into equation (48)

(49) $$Y' = \frac{f \; R \; (\cos \phi_c \sin \phi - \sin \phi_c \cos \phi \cos \lambda)}{R \; (1 - \cos \phi_c \cos \phi \cos \lambda - \sin \phi_c \sin \phi) + h}$$

In summary, equations (46) and (49) are perspective projection mapping equations derived for a spherical datum surface. These equations allow the calculation of plane X'-Y' coordinates of a point P' on a projection plane located at a distance f from the projection center. Initially, point P had been located on the datum sphere by spherical ϕ-λ coordinates and C by both spherical ϕ_c-λ_c coordinates and a distance h above the datum surface.

4.12 A Comparison Of Mapping Equations

Two sets of mapping equations have now been developed for the case of a projection center being placed at an arbitrary distance above a spherical datum surface. If both sets of equations are valid, they must prove to be equal. A brief comparison of equations (25) and (26), derived by the descriptive geometry approach of Chapter 3, with equations (46) and (49), just derived, illustrates a great similarity in the equations. The differences in the equations are due to the fact that while equations (46) and (49) contain the parameters h and f, equations

(25) and (26) contain the comparable parameter θ. In effect, the parameter θ fixes a f for a particular h. As explained earlier, however, since f acts only as a scale factor, its value is not of real significance.

Considering Figure 11 and recalling that AB was the projection plane, it is evident that DC, being the distance between the projection center and the projection plane, is the parameter f introduced in Section 4.11. From Figure 11, the following expression for the parameter f may be derived:

$$f = DC_F = O_FC_F - O_FD$$

$$= R + h - R \cos \theta$$

$$= R + h - \frac{R^2}{R + h} \tag{50}$$

Utilizing the definition of the parameter θ as given by equation (19) and the expression for f as given by equation (50), the perspective projection mapping equations derived by the descriptive geometry method of Chapter 3 will be proved equal to the mapping equations derived by the photogrammetric method of Section 4.11. Equation (26), derived in Chapter 3, may be transformed into equation (46), derived in Section 4.11, as follows:

Rewriting equation (26)

$$X' = \frac{R \sin^2 \theta \cos \phi \sin \lambda}{1 - \cos \theta \,(\cos \phi_c \cos \phi \cos \lambda + \sin \phi_c \sin \phi)}$$

Substituting $(1 - \cos^2 \theta)$ for $\sin^2 \theta$ and equation (19) into the above equation

$$X' = \frac{\left[R + h - \dfrac{R^2}{R + h}\right] R \cos \phi \sin \lambda}{R\,(1 - \cos \phi_c \cos \phi \cos \lambda - \sin \phi_c \sin \phi) + h}$$

Substituting equation (50) into the above equation

(51) $$X' = \frac{f\, R \cos \phi \sin \lambda}{R\,(1 - \cos \phi_c \cos \phi \cos \lambda - \sin \phi_c \sin \phi) + h}$$

In a like manner, equation (25), derived in Chapter 3, may be transformed into equation (49), derived in Section 4.11, as follows:

Rewriting equation (25)

$$Y' = \frac{R \sin^2 \theta\,(\cos \phi_c \sin \phi - \sin \phi_c \cos \phi \cos \lambda)}{1 - \cos \theta\,(\cos \phi_c \cos \phi \cos \lambda + \sin \phi_c \sin \phi)}$$

Substituting $(1 - \cos^2 \theta)$ for $\sin^2 \theta$ and equation (19) into the above equation

$$Y' = \frac{\left(R + h - \dfrac{R^2}{R + h}\right) R\,(\cos \phi_c \sin \phi - \sin \phi_c \cos \phi \cos \lambda)}{R\,(1 - \cos \phi_c \cos \phi \cos \lambda - \sin \phi_c \sin \phi) + h}$$

Substituting equation (50) into the above equation

(52) $$Y' = \frac{f\, R\,(\cos \phi_c \sin \phi - \sin \phi_c \cos \phi \cos \lambda)}{R\,(1 - \cos \phi_c \cos \phi \cos \lambda - \sin \phi_c \sin \phi) + h}$$

Equations (51) and (52) are seen to be identical to equations (46) and (49). Thus, the perspective projection mapping equations derived by the descriptive geometry method are, in effect, equal to the equations derived by the photogrammetric method. This equality is a partial proof of the validity of the derivations and the derived equations of Section 4.11.

4.13 The Gnomonic Projection

Three special cases of the perspective projection were discussed in Chapter 2, and three sets of mapping equations were developed for these special cases. Since the perspective projection mapping equations, equations (46) and (49), are applicable to all perspective projections of a spherical datum surface, these equations must be applicable also to the above mentioned special cases. This fact will be proved in this and the two succeeding sections.

As was illustrated in Section 2.1, the Gnomonic Projection is a special case of the perspective projection in which the projection center is located at the center of the sphere. By reviewing Figure 2 and by recalling the definitions given earlier for h and f, the values for h and f in the Gnomonic Projection are seen to be -R and -R respectively. Substituting these values into the perspective projection mapping equations, equations (46) and (49), yields the following:

$$X' = \frac{-R\,R\cos\phi\sin\lambda}{R(1-\cos\phi_c\cos\phi\cos\lambda-\sin\phi_c\sin\phi)-R}$$

$$= \frac{R\cos\phi\sin\lambda}{\cos\phi_c\cos\phi\cos\lambda+\sin\phi_c\sin\phi} \tag{53}$$

$$Y' = \frac{-R\,R(\cos\phi_c\sin\phi-\sin\phi_c\cos\phi\cos\lambda)}{R(1-\cos\phi_c\cos\phi\cos\lambda-\sin\phi_c\sin\phi)-R}$$

$$= \frac{R(\cos\phi_c\sin\phi-\sin\phi_c\cos\phi\cos\lambda)}{\cos\phi_c\cos\phi\cos\lambda+\sin\phi_c\sin\phi} \tag{54}$$

Equations (53) and (54) are identical to equations (7) and (8) developed for the Gnomonic Projection in Section 2.1. Thus, the mapping equations for the Gnomonic Projection have been proved to be special cases of the perspective projection mapping equations.

4.14 The Stereographic Projection

The Sterographic Projection was the second special case of the perspective projection discussed in Chapter 2. Considering Figure 5, the values for h and f in the Stereographic Projection are seen to be -2R and -2R respectively. Substituting these values into the perspective projection mapping equations, equations (46) and (49), yields the following:

$$X' = \frac{-2R\ R \cos\phi \sin\lambda}{R\ (1 - \cos\phi_c \cos\phi \cos\lambda - \sin\phi_c \sin\phi) - 2R}$$

$$(55) \qquad = \frac{2R \cos\phi \sin\lambda}{1 + \cos\phi_c \cos\phi \cos\lambda + \sin\phi_c \sin\phi}$$

$$Y' = \frac{-2R\ R\ (\cos\phi_c \sin\phi - \sin\phi_c \cos\phi \cos\lambda)}{R\ (1 - \cos\phi_c \cos\phi \cos\lambda + \sin\phi_c \sin\phi) - 2R}$$

$$(56) \qquad = \frac{2R\ (\cos\phi_c \sin\phi - \sin\phi_c \cos\phi \cos\lambda)}{1 + \cos\phi_c \cos\phi \cos\lambda + \sin\phi_c \sin\phi}$$

Equations (55) and (56) are identical to equations (12) and (13) developed for the Stereographic Projection in Section 2.2. Thus, the equations for the Stereographic Projection have been proved to be special cases of the perspective projection mapping equations.

4.15 The Orthographic Projection

The third special case of the perspective projection discussed in Chapter 2 was the Orthographic Projection. In order to convert the perspective projection mapping equations for use in an Orthographic Projection, a relationship must be established between f and h. It was discussed earlier that f has the effect of a scale factor, and from elementary projection principles it was seen that a full scale projection results if f equals h. Therefore, in developing an Orthographic Projection, f may be set equal to h. Making this substitution in the perspective projection mapping equations, equations (46) and (49), yields the following:

$$X' = \frac{h\,R \cos\phi \sin\lambda}{R\,(1 - \cos\phi_c \cos\phi \cos\lambda - \sin\phi_c \sin\phi) + h}$$

$$Y' = \frac{h\,R\,(\cos\phi_c \sin\phi - \sin\phi_c \cos\phi \cos\lambda)}{R\,(1 - \cos\phi_c \cos\phi \cos\lambda - \sin\phi_c \sin\phi) + h}$$

The numerators and denominators of the above equations are now divided by h, producing:

$$X' = \frac{R \cos\phi \sin\lambda}{\dfrac{R\,(1 - \cos\phi_c \cos\phi \cos\lambda - \sin\phi_c \sin\phi)}{h} + 1}$$

$$Y' = \frac{R\,(\cos\phi_c \sin\phi - \sin\phi_c \cos\phi \cos\lambda)}{\dfrac{R\,(1 - \cos\phi_c \cos\phi \cos\lambda - \sin\phi_c \sin\phi)}{h} + 1}$$

The Orthographic Projection is based on the projection center being located at infinity. Substituting infinity for h in the above equations yields:

(57) $$X' = R \cos \phi \sin \lambda$$

(58) $$Y' = R (\cos \phi_c \sin \phi - \sin \phi_c \cos \phi \cos \lambda)$$

Equations (57) and (58) are identical to equations (17) and (18) developed for the Orthographic Projection in Section 2.3. Thus, the equations for the Orthographic Projection have also been proved to be special cases of the perspective projection mapping equations.

4.2 An Ellipsoidal Datum Surface

4.21 The General Mapping Equations

Section 4.1 dealt with perspective projection mapping equations for a spherical datum surface; however, the sphere is actually a rather poor approximation for the shape of the earth. A better approximation for the shape of the earth is an ellipsoid of revolution, and this section deals with the derivation of General Perspective Projection Mapping Equations for ellipsoidal datum surfaces. In the remainder of this paper, an ellipsoid of revolution will be implied by the term ellipsoid. The parameters used to define the ellipsoid will be a, the major semi-diameter, and e^2, the square of the eccentricity. It will be seen that the derivation method to be used in this section is analogous to the method employed in Section 4.11.

Figure 14 illustrates the datum surface which is an ellipsoid of revolution with the Z axis being the axis of rotation. Point C depicts the projection center and is located at a geodetic latitude

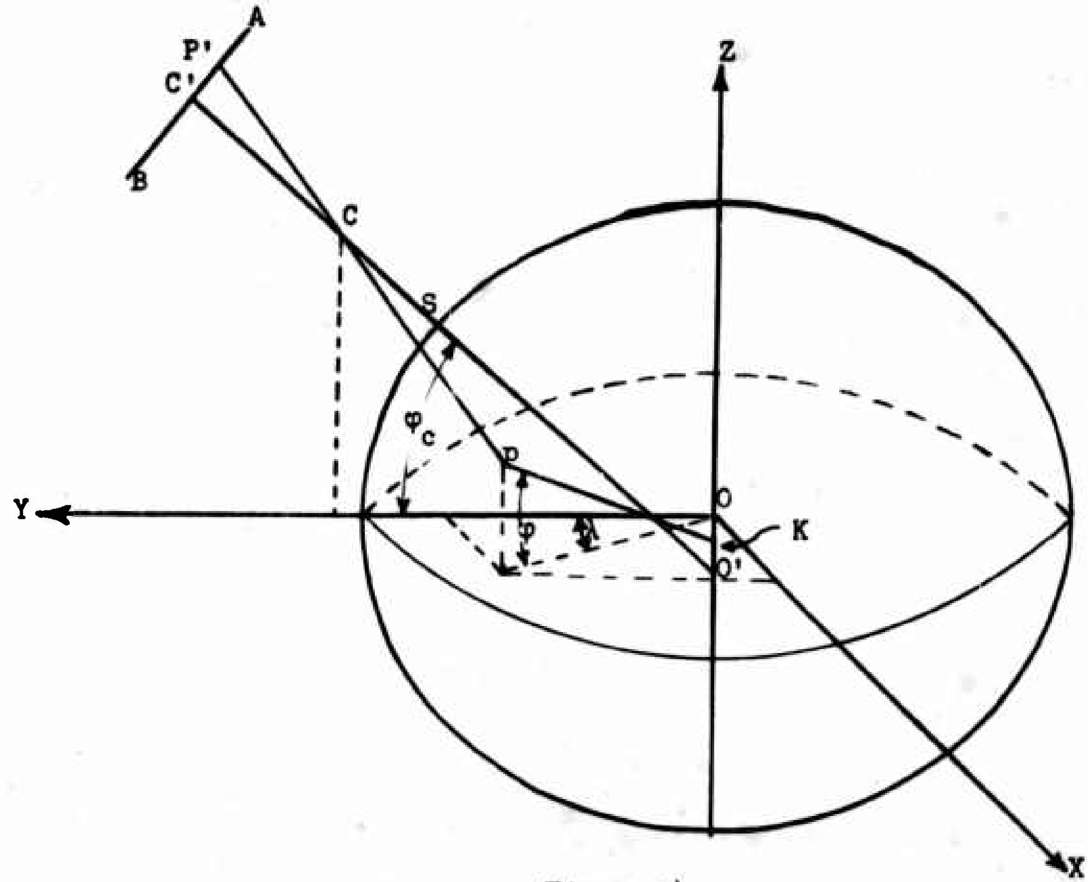

Figure 14

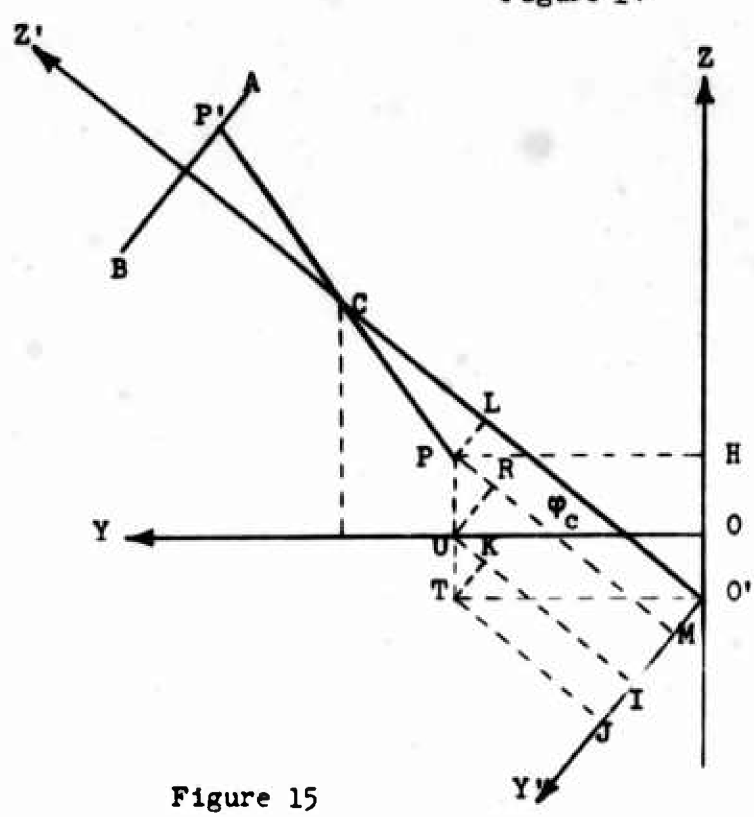

Figure 15

φ_c, at a longitude λ_c, and at a height above the datum surface of h. Point P represents an arbitrary point on the datum surface and is located at a geodetic latitude φ and at a longitude λ. As before, λ will equal the difference in longitude between the point P and the projection center. N_c and N_P are the radii of curvature in the prime vertical at C and P respectively.

Figure 14 illustrates the following:

$CS = h$

$CC' = f$

$KP = N_P$

$O'S = N_c$

The space rectangular coordinates of point C are obtained from Figure 14.

(59) $$X_c = 0$$

(60) $$Y_c = (N_c + h) \cos \varphi_c$$

(61) $$Z_c = [N_c (1 - e^2) + h] \sin \varphi_c$$

Likewise, the coordinates of point P are obtained from Figure 14.

(62) $$X_p = N_p \cos \varphi \sin \lambda$$

(63) $$Y_p = N_p \cos \varphi \cos \lambda$$

(64) $$Z_p = N_p (1 - e^2) \sin \varphi$$

Similar to the derivation utilizing a spherical datum in Section 4.11, the Z axis is now rotated, and in this case

transposed, in the Z-Y plane until the Z axis coincides with the projection axis CO'. The new X'-Y'-Z' coordinate system is illustrated in Figure 15. The values of X coordinates of points are not changed by the rotation due to the rotation being accomplished in the Z-Y plane.

Figure 15 illustrates the following:

$$PH = Y_p$$

$$PU = Z_p$$

$$PL = Y_p'$$

$$PM = Z_p'$$

$$JU = Y_p \sin \varphi_c$$

$$TK = OO' \cos \varphi_c$$

$$UR = Z_p \cos \varphi_c$$

$$JT = Y_p \cos \varphi_c$$

$$UK = OO' \sin \varphi_c$$

$$RP = Z_p \sin \varphi_c$$

The distance OO' which the Z' axis has been transposed along the Z axis is computed as follows:

$$OO' = (N_c + h) \sin \varphi_c - Z_c$$

$$= (N_c + h) \sin \varphi_c - [N_c (1 - e^2) + h] \sin \varphi_c$$

$$= N_c e^2 \sin \varphi_c \tag{65}$$

The space rectangular coordinates of C in the new X'-Y'-Z' coordinate system are obtained from Figure 15.

(65a) $$X_c' = 0$$

(66) $$Y_c' = 0$$

(67) $$Z_c' = N_c + h$$

Likewise, the new coordinates of point P are obtained.

(68) $$X_p' = X_p$$

(69) $$Y_p' = Y_p \sin \varphi_c - (Z_p + OO') \cos \varphi_c$$

(70) $$Z_p' = Y_p \cos \varphi_c + (Z_p + OO') \sin \varphi_c$$

Substituting equations (62), (63), (64), and (65) into equations (68), (69), and (70), the coordinates of point P are obtained in terms of the original parameters.

(71) $$X_p' = N_p \cos \varphi \sin \lambda$$

(72) $$Y_p' = N_p \sin \varphi_c \cos \varphi \cos \lambda - N_p (1 - e^2) \cos \varphi_c \sin \varphi - N_c e^2 \sin \varphi_c \cos \varphi_c$$

(73) $$Z_p' = N_p \cos \varphi_c \cos \varphi \cos \lambda + N_p (1 - e^2) \sin \varphi_c \sin \varphi + N_c e^2 \sin^2 \varphi_c$$

The equation of the projection plane AB which is perpendicular to the projection axis and hence now perpendicular to the Z' axis is now formulated as

(74) $$Z' = N_c + h + f$$

The equation of the projection ray from P that passes through C and intersects the projection plane at P' is now written as

$$(75) \qquad \frac{X' - X_p'}{X_c' - X_p'} = \frac{Y' - Y_p'}{Y_c' - Y_p'} = \frac{Z' - Z_p'}{Z_c' - Z_p'}$$

Similar to the derivation in Section 4.11, the simultaneous solution of equations (75) and (74) will yield the desired X'-Y' coordinates of the projected point on the projection plane.

The equation for X' is derived in the following manner:

From equation (75)

$$(76) \qquad X' = \frac{(Z' - Z_p')(X_c' - X_p')}{(Z_c' - Z_p')} + X_p'$$

Substituting equation (65a) into equation (76)

$$(77) \qquad X' = \frac{X_p'(Z_c' - Z')}{Z_c' - Z_p'}$$

Substituting equations (67), (71), (73), and (74) into equation (77)

$$X' = \frac{-f \; N_p \cos\varphi \sin\lambda}{N_c(1 - e^2 \sin^2\varphi_c) - N_p[\cos\varphi_c \cos\varphi\cos\lambda + (1 - e^2)\sin\varphi_c \sin\varphi] + h}$$

As in Section 4.11, the sign of the above equation must be reversed to compensate for the inversion due to the photogrammetric derivation, and therefore

$$X' = \frac{f\, N_p \cos\varphi \sin\lambda}{N_c\,(1 - e^2 \sin^2 \varphi_c) - N_p\,[\cos\varphi_c \cos\varphi \cos\lambda + (1 - e^2) \sin\varphi_c \sin\varphi] + h} \tag{78}$$

The equation for Y' is derived in a manner analogous to the derivation of the equation for X' as follows:

From equation (75)

$$Y' = \frac{(Z' - Z_p')\,(Y_c' - Y_p')}{(Z_c' - Z_p')} + Y_p' \tag{79}$$

Substituting equation (66) into equation (79)

$$Y' = \frac{Y_p'\,(Z_c' - Z')}{Z_c' - Z_p'} \tag{80}$$

Substituting equations (67), (72), (73), and (74) into equation (80)

$$Y' = \frac{f\,[N_c\, e^2 \sin\varphi_c \cos\varphi_c + N_p\,[(1 - e^2) \cos\varphi_c \sin\varphi - \sin\varphi_c \cos\varphi \cos\lambda]\,]}{N_c\,(1 - e^2 \sin^2 \varphi_c) - N_p\,[\cos\varphi_c \cos\varphi \cos\lambda + (1 - e^2) \sin\varphi_c \sin\varphi] + h} \tag{81}$$

In summary, equations (78) and (81) are General Perspective Projection Mapping Equations derived for an ellipsoidal datum surface. These equations allow the calculation of plane X'-Y' coordinates of a point P' on a projection plane located at a distance

f from the projection center. Initially, point P had been located on the datum ellipsoid by geodetic φ-λ coordinates and C by both geodetic φ_c-λ_c coordinates and a distance h above the datum surface.

4.22 A Comparison of Mapping Equations For The Ellipsoid And For The Sphere

A sphere is actually a special case of an ellipsoid in which the ellipsoid flattening is equal to zero. Since equations (78) and (81) were derived as General Perspective Projection Mapping Equations for an ellipsoidal datum surface, these equations should prove to be suitable for a spherical datum surface. Therefore, equations (46) and (49), derived for a spherical datum surface, should prove to be special cases of equations (78) and (81). This is illustrated as follows:

Rewriting equations (78) and (81)

$$X' = \frac{f\, N_p \cos\varphi \sin\lambda}{N_c(1 - e^2 \sin^2\varphi_c) - N_p[\cos\varphi_c \cos\varphi \cos\lambda + (1 - e^2)\sin\varphi_c \sin\varphi] + h}$$

$$Y' = \frac{f\,[N_c e^2 \sin\varphi_c \cos\varphi_c + N_p[(1 - e^2)\cos\varphi_c \sin\varphi - \sin\varphi_c \cos\varphi \cos\lambda]\,]}{N_c(1 - e^2 \sin^2\varphi_c) - N_p[\cos\varphi_c \cos\varphi \cos\lambda + (1 - e^2)\sin\varphi_c \sin\varphi] + h}$$

Considering the parameters in the above equations, it is apparent the following is true when the datum surface is spherical:

$$N_c = N_p = R$$

$$e = 0$$

The substitution of the above equalities into equations (78) and (81) transforms these equations into equations (46) and (49), the mapping equations for a spherical datum. Thus, equations (78) and (81) are valid both for spherical and ellipsoidal datum surfaces. Further, since it was shown that the mapping equations for the Gnomonic, Stereographic, and Orthographic Projections were only special cases of the mapping equations for a spherical datum, then they too are only special cases of equations (78) and (81). Therefore, equations (78) and (81) are truly General Perspective Projection Mapping Equations.

5. AN EMPIRICAL DETERMINATION OF PERSPECTIVE MAP PROJECTION DISTORTIONS

Map distortions is an extensive subject and no attempt will be made in this paper to treat the subject in general. The author has derived an empirical method to calculate the linear distortions of the meridians and parallels on the projection plane and the angular distortions of the intersections of the meridians and parallels on the projection plane. The calculation of these distortions not only enables a vizualization of the projected latitude-longitude grid but also may be used as indicators as to the type of distortions that will be found in projecting any line from the datum surface onto the projection plane.

5.1 Meridian And Parallel Linear Distortions

The author defines a linear distortion to exist when the linear distance between two points measured on the datum surface does not equal the linear distance between the same two points projected full scale onto the projection plane. The amount of distortion will be indicated by a ratio of the distance measured on the projection plane to the distance measured on the datum surface. In order to determine the linear distortions at a point, the two distances to be compared would of necessity have to be infinitely small. However, as will be shown by examples in Chapter 6, good approximations can be made by utilizing finite distances.

The distance between two points, P_1 and P_2, located by plane X'-Y' coordinates on a projection plane is given by the following elementary equation:

$$d' = [(X_2' - X_1')^2 + (Y_2' - Y_1')^2]^{\frac{1}{2}} \tag{82}$$

The distance between two points on the same parallel on the projection plane, d_p', would thus be obtained by using equations (78) and (81) to compute the X'-Y' coordinates of the two points and then by using equation (82) to calculate the distance between the points. Likewise, the distance between two points on the same meridian on the projection plane, d_m', would also be obtained by using equations (78), (81), and (82).

The distance between two points on a Meridian on the datum surface is calculated by the following formula as presented in [6]:

$$\begin{aligned} d_m = a\,(1 - e^2)\,[A\ (\varphi_2 - \varphi_1) - \frac{B}{2}\,(\sin 2\varphi_2 - \sin 2\varphi_1) \\ + \frac{C}{4}\,(\sin 4\varphi_2 - \sin 4\varphi_1) - \frac{D}{6}\,(\sin 6\varphi_2 - \sin 6\varphi_1) \\ + \frac{E}{8}\,(\sin 8\varphi_2 - \sin 8\varphi_1) - \frac{F}{10}\,(\sin 10\varphi_2 - \sin 10\varphi_1) \\ + \ \cdots\cdots\cdots] \end{aligned} \tag{83}$$

The parameters in the above equation have the following definitions:

a = ellipsoid major semi-diameter

e = ellipsoid eccentricity

$$A = 1 + \frac{3}{4} e^2 + \frac{45}{64} e^4 + \frac{175}{256} e^6 + \frac{11025}{16384} e^8 + \frac{43659}{65536} e^{10} + \ldots$$

$$B = \frac{3}{4} e^2 + \frac{15}{16} e^4 + \frac{525}{512} e^6 + \frac{2205}{2048} e^8 + \frac{72765}{65536} e^{10} + \ldots$$

$$C = \frac{15}{64} e^4 + \frac{105}{256} e^6 + \frac{2205}{4096} e^8 + \frac{10395}{16384} e^{10} + \ldots$$

$$D = \frac{35}{512} e^6 + \frac{35}{2048} e^8 + \frac{31185}{131072} e^{10} + \ldots$$

$$E = \frac{315}{16384} e^8 + \frac{3465}{65536} e^{10} + \ldots$$

$$F = \frac{693}{131072} e^{10} + \ldots$$

The distance between two points on a parallel on the datum surface is calculated by the following formula:

(84) $$d_p = N \cos \varphi \, (\lambda_2 - \lambda_1)$$

where N is the radius of curvature in the prime vertical.

Utilizing equations (82), (83), and (84), the linear distortions along meridians and parallels at a particular point are calculated by the following ratios:

(85) $$\text{Meridian Distortion} = \frac{d_m'}{d_m}$$

(86) $$\text{Parallel Distortion} = \frac{d_p'}{d_p}$$

Equations (85) and (86) represent ratios of projection plane distances to datum surface distances. By making the distances that are compared small, the above equations offer a good approximation of the distortions at a particular point.

5.2 Angular Distortions

The meridians and parallels of a sphere or an ellipsoid of revolution intersect at right angles on the datum surface. When these angles are projected onto the projection plane, they may not intersect at right angles and hence, distortions exist. The author chooses to indicate this distortion by the absolute value of the number of degrees that the projected angle differs from ninety degrees.

An angle on the projection plane can be calculated from a knowledge of the coordinates of three points.

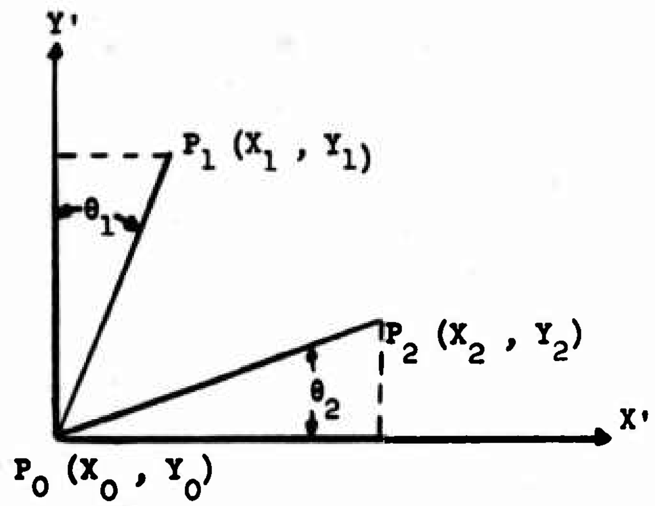

Figure 16

Figure 16 depicts a projection plane on which a meridian, a segment of which is P_1P_0, and a parallel, a segment of which is P_2P_0, intersect at P_0. From Figure 16, it is evident that by knowing the coordinates of P_0, P_1, and P_2 in a plane coordinate system, the angles θ_1 and θ_2 may be calculated by elementary geometry. In the example illustrated by Figure 16, the angular distortion, as previously defined, has the value $(\theta_1 + \theta_2)$. Naturally the points P_1 and P_2 could be located in other quadrants depending on the projection, but the basic principles in determining the angular distortion remains the same.

Usually the meridians and parallels are projected onto the projection plane as some type of curved lines. Thus, the segment of meridian P_1P_0 and segment of paralled P_2P_0 as illustrated in Figure 16 are actually curved lines. In order to obtain exact angular values, the points P_1 and P_2 would have to be taken infinitely close to P_0. However, it will be illustrated in Chapter 6 that good approximations can be obtained for angular distortions by using finite distances for P_1P_0 and P_2P_0.

5.3 Distortion Calculations

The methods discussed for distortion calculation in Sections 5.1 and 5.2 indicate that basically the same quantities are required to calculate both linear and angular distortions at a particular point. In the calculation of both types of distortions, the plane X'-Y' coordinates of three points are required. Since the calculation of plane coordinates is a rather lengthy procedure, the distortion

calculation methods discussed in this paper are especially suited for use when electronic computers are available.

6. EXAMPLES OF PERSPECTIVE MAP PROJECTION GRIDS

6.1 General Format Of The Projection Grids

This paper has dealt with the derivation of General Perspective Projection Mapping Equations and with the empirical derivation of the distortions inherent with the perspective projection. In this chapter six examples of the perspective projection are illustrated and briefly discussed. These examples are included both to illustrate various types of perspective projections and to partially verify the validity of the equations derived in prior chapters. The coordinates for all the examples were computed through the use of equations (78) and (81). All calculations were carried out on the IBM 7094 computer. The Appendix includes the computer program used to produce one of the examples and also includes a brief discussion of the program.

A similar format is used for all the included examples. The first page of each example lists all the projection parameters. The body of each example is a numerical grid which tabulates the coordinates and distortions of the intersections of meridians and parallels spaced ten degrees apart. The grid is read by selecting the longitude of a particular desired meridian from the top line and then by reading down the column under that longitude until the desired latitude, as indicated by the column on the left of the grid, is reached. For each intersection of a meridian and a parallel, five quantities, as listed on the left of the grid,

are tabulated. The first two quantities are the X' and Y' coordinates of the intersection as computed by equations (78) and (81). The next two quantities are the meridian distortion and the parallel distortion computed at the intersection point by equations (85) and (86) respectively. The fifth quantity is the angular distortion of the intersection of the meridian and the parallel, calculated as described in Section 5.2.

Depending on the type of projection, the grid illustrates either one half of the datum surface visible from the projection center or a grid 90 degrees in longitude and 170 degrees in latitude. In both of the above cases the first column of the body of the grid is the central meridian of the projection. The remaining columns describe meridians to the right of the central meridian. The grid to the left of the central meridian, if calculated, would be symmetrical to that calculated to the right of the central meridian. The central latitude is found in the center of the grid and may be located by the latitude indicators to the left of the grid. An inspection of these latitude indicators reveals that, as would be expected, the absolute values of the latitudes increase until a geographic pole is reached and then they proceed to decrease. It is evident that as a particular meridian passes through a pole, the meridian undergoes a 180 degree change in longitude and after passing through the pole the meridian is actually to the left of the central meridian. Being that only the grid to the right of the central meridian is being considered, as a meridian passes over a

pole the symmetric meridian to the right of the central meridian is considered rather than the original meridian. The above can be clarified by the following specific example of a meridian passing over a pole. In the examples illustrated in this chapter, the 90 Degrees West Longitude Meridian is the central meridian. Thus, as the 80 Degrees West Longitude Meridian passes through a pole, it assumes a longitude of 100 degrees East and is now located to the left of the central meridian. To complete the right side of the grid, the meridian symmetric to the 100 Degrees East Longitude Meridian is considered. This symmetric meridian is the 80 Degrees East Longitude Meridian. Therefore, the longitude indicators located at the top of the grids are numerically correct but may indicate East or West Longitude. It should be kept in mind when viewing the grid that the actual value of a meridian in no way enters into the computation of the plane X'-Y' coordinates of points on that meridian. It is the longitude difference between a particular point and the central meridian that determines the values of the coordinates of that point.

It will be noted that the distortions listed for 90 degrees of latitude are numbers consisting of a series of ones. These numbers are not the distortions at this latitude but merely indicators that the distortions at this latitude were not calculated. It is apparent from the distortion analysis in Chapter 5 that the equations used to compute distortions for the rest of the grid will not function properly at the pole. No effort is made in this paper to establish special distortion equations for the poles.

Earlier in this paper it was stated that to calculate exact distortions an infinitely small distance between two points on the datum surface and the corresponding distance on the projection plane would have to be compared. The distance actually selected would determine the accuracy of the calculations. Various small distances were selected and tested by calculating distortions at points where distortions were known quantities. It was determined that by using a distance of .00001 degrees of arc, or approximately 1 meter on the surface of the earth, errors were apparently eliminated in the values printed out in the examples. The manner in which the length of the distance affected the accuracy of the distortion computations varied according to the type of projection.

6.2 Sphere And Ellipsoid Parameters

The sphere that was selected to represent the datum surface in the examples is only one of many possible spherical approximations of the shape of the earth. A sphere radius of 6,371,224 meters was selected. This radius produces a sphere with approximately the same area and volume as the International Ellipsoid.

The International Ellipsoid was selected to represent the ellipsoidal datum surface approximating the shape of the earth. The International Ellipsoid is defined as an ellipsoid with a semi-major axis equal to 6,378,388 meters and a flattening equal to $\frac{1}{297}$.

6.3 Example 1 - The Gnomonic Projection

The first example illustrates a Gnomonic Projection. Since the projection center is located at the center of the sphere in a Gnomonic Projection, it is obvious that a complete hemisphere can not be projected onto a plane. The computer was programmed not to print out coordinates larger than the absolute value of 99,999,999, and this is the reason that a complete square grid was not printed. It can be seen that both the size of the coordinates and the size of the distortions increase rapidly as the location of points progress away from the central meridian and the central parallel. The property of a Gnomonic Projection that great circles on the datum surface are projected as straight lines is clearly illustrated by the projection of the equator. It is seen that the equator, the parallel with 0 degrees of latitude, has a constant Y' coordinate, readily identifying it as a straight line.

The following five pages illustrate a Gnomonic Projection.

PERSPECTIVE MAP PROJECTIONS

EXAMPLE 1 - GNOMONIC PROJECTION

PROJECTION PARAMETERS -

CENTRAL LATITUDE = 40 DEGREES

CENTRAL LONGITUDE = 90 DEGREES

HEIGHT OF PROJECTION CENTER = -6371224.000 METERS

FOCAL LENGTH (SCALE FACTOR) = -6371224.000 METERS

SPHERE RADIUS = 6371224.000 METERS

LONGITUDE.....	90	80	70	60	50
LAT. 60					
X	-0.00	3082323.04	5537782.64	7080288.71	7778184.00
Y	36133006.84	34788105.78	31263162.51	26671423.45	22013989.15
MER.	33.1634	30.7718	24.9310	18.2526	12.5585
PAR.	5.7588	6.9358	8.4643	8.7772	8.0566
ANG.	0.0000	35.8472	51.5219	56.7526	57.2000
LAT. 70					
X	-0.00	1093623.91	2082862.58	2889102.03	3472925.00
Y	17504794.07	17241916.27	16495742.14	15377882.36	14031869.08
MER.	8.5486	8.2788	7.5375	6.4951	5.3507
PAR.	2.9238	3.0964	3.4638	3.7795	3.9129
ANG.	0.0000	19.6499	33.2758	40.5782	43.2785
LAT. 80					
X	-0.00	382685.08	744838.44	1068274.71	1338957.81
Y	11035283.67	10969339.75	10776604.00	10471494.34	10075409.70
MER.	4.0000	3.9326	3.7393	3.4450	3.0854
PAR.	2.0000	2.0607	2.2148	2.4021	2.5675
ANG.	0.0000	12.7587	22.9367	29.5021	32.5893
LAT. 90					
X	-0.00	-0.00	-0.00	-0.00	-0.00
Y	7592929.09	7592929.09	7592929.09	7592929.09	7592929.09
MER.	1.1111	1.1111	1.1111	1.1111	1.1111
PAR.	1.1111	1.1111	1.1111	1.1111	1.1111
ANG.	1.1111	1.1111	1.1111	1.1111	1.1111
LAT. 80					
X	0.00	251452.87	499186.52	739319.47	967650.82
Y	5346091.71	5374373.43	5459248.11	5600764.23	5798866.28
MER.	1.7041	1.6979	1.6796	1.6500	1.6115
PAR.	1.3054	1.3286	1.3958	1.5014	1.6379
ANG.	0.0000	6.2727	11.6647	15.4992	17.3497
LAT. 70					
X	0.00	438949.80	876581.94	1311243.70	1740574.87
Y	3678427.89	3720097.73	3846140.79	4059661.39	4365834.59
MER.	1.3333	1.3337	1.3350	1.3379	1.3440
PAR.	1.1547	1.1738	1.2307	1.3252	1.4568
ANG.	0.0000	4.3158	8.0090	10.5380	11.4693
LAT. 60					
X	0.00	592345.35	1188687.69	1792938.51	2408792.16
Y	2318935.89	2366696.69	2512102.19	2761690.06	3126933.38
MER.	1.1325	1.1364	1.1487	1.1765	1.2044
PAR.	1.0642	1.0816	1.1347	1.2258	1.3595
ANG.	0.0000	2.7071	4.9340	6.2286	6.1759
LAT. 50					
X	0.00	727646.91	1466518.95	2228544.75	3027155.82
Y	1123418.69	1172938.05	1324563.56	1587908.64	1980462.76
MER.	1.0311	1.0376	1.0579	1.0941	1.1509
PAR.	1.0154	1.0326	1.0857	1.1791	1.3215
ANG.	0.0000	1.2844	2.1720	2.2685	1.1794

LONGITUDE.....	90	80	70	60	50
LAT. 40					
X	0.00	855138.08	1730520.25	2648547.94	3636468.97
Y	0.00	48090.08	196138.55	456171.12	850772.12
MER.	1.0000	1.0090	1.0372	1.0881	1.1694
PAR.	1.0000	1.0180	1.0742	1.1754	1.3347
ANG.	0.0000	0.0575	0.4621	1.5709	3.7619
LAT. 30					
X	0.00	982969.06	1997403.32	3079291.61	4275167.26
Y	-1123418.69	-1079756.03	-944604.05	-704507.64	-333399.71
MER.	1.0311	1.0432	1.0811	1.1508	1.2646
PAR.	1.0154	1.0354	1.0984	1.2139	1.4017
ANG.	0.0000	1.4030	3.1230	5.4935	8.8810
LAT. 20					
X	0.00	1119378.70	2284632.71	3549944.57	4989577.74
Y	-2318935.89	-2283291.21	-2172313.25	-1972725.67	-1657944.92
MER.	1.1325	1.1490	1.2013	1.2991	1.4631
PAR.	1.0642	1.0877	1.1624	1.3025	1.5382
ANG.	0.0000	2.8370	5.9714	9.7202	14.4393
LAT. 10					
X	0.00	1274969.60	2615363.25	4101181.25	5849113.43
Y	-3678427.89	-3656061.74	-3585960.12	-3458084.00	-3251558.01
MER.	1.3333	1.3572	1.4334	1.5786	1.8306
PAR.	1.1547	1.1843	1.2793	1.4620	1.7821
ANG.	0.0000	4.4683	9.2140	14.5356	20.7709
LAT. 0					
X	0.00	1466518.95	3027155.82	4801846.58	6978827.09
Y	-5346091.71	-5346091.71	-5346091.71	-5346091.71	-5346091.71
MER.	1.7041	1.7415	1.8624	2.0988	2.5275
PAR.	1.3054	1.3460	1.4783	1.7405	2.2245
ANG.	0.0000	6.4664	13.1678	20.3606	28.3408
LAT. -10					
X	0.00	1725800.37	3592855.96	5791251.58	8649394.66
Y	-7592929.09	-7633718.28	-7764072.95	-8012135.21	-8443389.92
MER.	2.4203	2.4867	2.7051	3.1478	4.0030
PAR.	1.5557	1.6187	1.8282	2.2608	3.1190
ANG.	0.0000	9.1451	18.3924	27.8803	37.8144
LAT. -20					
X	0.00	2125755.44	4484726.10	7417743.06	11605863.02
Y	-11035283.67	-11162501.15	-11576207.83	-12394867.28	-13924798.91
MER.	4.0000	4.1437	4.6292	5.6720	7.9160
PAR.	2.0000	2.1190	2.5251	3.4164	5.3992
ANG.	0.0000	13.1958	26.0069	38.2607	50.0770
LAT. -30					
X	0.00	2886438.59	6248587.75	10898470.12	18985371.82
Y	-17504794.07	-17873969.19	-19115510.04	-21773874.31	-27606699.42
MER.	8.5486	8.9950	10.5806	14.4155	24.9401
PAR.	2.9238	3.2455	4.3835	7.1647	15.0910
ANG.	0.0000	20.5179	38.4906	53.3529	65.9492
LAT. -40					
X	0.00	5144775.88	12073609.93	25679836.37	
Y	-36133006.84	-37799162.26	-44013496.05	-61603770.81	
MER.	33.1634	36.5226	50.4864	102.2915	
PAR.	5.7588	7.8025	15.4977	43.7005	
ANG.	0.0000	38.2803	61.0794	74.8895	

LONGITUDE.....	40	30	20	10	0
LAT. 60					
X	7860123.26	7555116.37	7032448.92	6400399.77	5722617.92
Y	17853591.29	14378916.26	11574963.03	9348661.70	7592929.09
MER.	8.4003	5.6116	3.8309	2.7321	2.0743
PAR.	6.8875	5.6792	4.6154	3.7446	3.0558
ANG.	50.3708	33.9839	16.2031	3.2201	23.8587
LAT. 70					
X	3832039.33	3989554.00	3980602.22	3842220.09	3607623.82
Y	12595301.16	11176336.80	9846894.01	8646911.87	7592929.09
MER.	4.2671	3.3442	2.6231	2.1042	1.7618
PAR.	3.8514	3.6415	3.3435	3.0090	2.6738
ANG.	42.5982	39.1155	25.9292	5.7041	15.5794
LAT. 80					
X	1547922.86	1691279.19	1769505.32	1786361.88	1747729.23
Y	9613598.88	9112031.95	8594888.77	8082956.89	7592929.09
MER.	2.7011	2.3315	2.0109	1.7645	1.6041
PAR.	2.6773	2.7183	2.6918	2.6078	2.4799
ANG.	32.5936	29.7884	24.3653	13.9754	7.6926
LAT. 90					
X	-0.00	-0.00	-0.00	-0.00	-0.00
Y	7592929.09	7592929.09	7592929.09	7592929.09	7592929.09
MER.	1.1111	1.1111	1.1111	1.1111	1.1111
PAR.	1.1111	1.1111	1.1111	1.1111	1.1111
ANG.	1.1111	1.1111	1.1111	1.1111	1.1111
LAT. 80					
X	1179515.96	1369668.04	1532206.24	1660582.41	1747729.23
Y	6053180.35	6362696.58	6725336.97	7137404.62	7592929.09
MER.	1.5684	1.5291	1.5077	1.5248	1.6041
PAR.	1.7969	1.9702	2.1487	2.3225	2.4799
ANG.	16.9554	14.1371	8.8287	1.2767	7.6926
LAT. 70					
X	2161059.30	2567459.60	2952097.25	3303954.99	3607623.82
Y	4771866.53	5286843.12	5921341.91	6686601.09	7592929.09
MER.	1.3571	1.3850	1.4427	1.5559	1.7618
PAR.	1.6258	1.8327	2.0777	2.3597	2.6738
ANG.	10.4537	7.2057	1.5667	6.2865	15.5794
LAT. 60					
X	3039487.42	3687368.48	4353088.87	5034192.04	5722617.92
Y	3625160.06	4280943.68	5128048.38	6211969.24	7592929.09
MER.	1.2561	1.3367	1.4678	1.6902	2.0743
PAR.	1.5428	1.7863	2.1052	2.5196	3.0558
ANG.	4.3903	0.5239	5.6490	14.0153	23.8587
LAT. 50					
X	3878300.50	4801846.58	5823655.68	6978827.09	8317042.25
Y	2530167.35	3279922.16	4295358.80	5678524.57	7592929.09
MER.	1.2374	1.3716	1.5896	1.9654	2.6511
PAR.	1.5273	1.8199	2.2377	2.8455	3.7559
ANG.	1.4949	6.1398	13.0316	22.1139	32.7324

LONGITUDE.....	40	30	20	10	0
LAT. 40					
X	4730372.32	5981928.84	7470988.40	9331603.41	11812500.70
Y	1417866.59	2219975.14	3362577.79	5033120.18	7592929.09
MER.	1.2962	1.4987	1.8419	2.4742	3.7653
PAR.	1.5758	1.9413	2.5104	3.4431	5.0984
ANG.	7.4403	13.0158	20.8027	30.7934	42.3941
LAT. 30					
X	5652046.28	7316509.42	9456387.15	12445913.44	17167853.75
Y	214766.67	1021258.11	2238371.23	4178814.84	7592929.09
MER.	1.4479	1.7542	2.3090	3.4436	6.2229
PAR.	1.6998	2.1825	3.0100	4.5727	8.0424
ANG.	13.6920	20.3511	29.1988	40.2634	52.9955
LAT. 20					
X	6719319.00	8943015.50	12071566.38	17097576.91	27232625.23
Y	-1178518.91	-439665.70	757559.56	2902788.73	7592929.09
MER.	1.7380	2.2261	3.1958	5.5198	13.2993
PAR.	1.9321	2.6211	3.9482	6.9752	16.4893
ANG.	20.5317	28.4251	38.4688	50.7024	64.5862
LAT. 10					
X	8057257.36	11116142.05	15950704.66	25468232.98	56212979.68
Y	-2925073.40	-2391562.69	-1438952.71	606583.12	7592929.09
MER.	2.2754	3.1315	5.0802	11.1511	51.5931
PAR.	2.3515	3.4487	5.9224	13.4624	62.1357
ANG.	28.3068	37.5549	48.8464	62.1979	77.0375
LAT. 0					
X	9911864.98	14405539.75	22850885.78	47168290.52	
Y	-5346091.71	-5346091.71	-5346091.71	-5346091.71	
MER.	3.3396	5.1004	10.1119	37.0958	
PAR.	3.1594	5.2216	11.1595	43.2918	
ANG.	37.4537	48.0699	60.4798	74.6602	
LAT. -10					
X	12875537.14	20459851.38	40272557.57		
Y	-9214890.95	-10784059.71	-15210889.76		
MER.	5.8104	10.6082	32.3848		
PAR.	4.9908	10.1311	34.2720		
ANG.	48.4801	60.2183	73.3143		
LAT. -20					
X	18884397.27	37014592.43			
Y	-17058900.76	-25653487.89			
MER.	13.7282	38.1342			
PAR.	10.7676	34.5563			
ANG.	61.8183	73.9726			
LAT. -30					
X	40239340.38				
Y	-44935799.15				
MER.	73.3868				
PAR.	54.0311				
ANG.	77.4316				

6.4 Example 2 - The Stereographic Projection

The primary property of a Stereographic Projection is that it is a conformal type projection. Conformal projections have two properties which are immediately apparent in the following example. First, angles are projected from the datum surface to the projection plane with true size. Thus, it is seen in the example that all angular distortions are equal to zero, or, in other words, all the projected meridians and parallels intersect at right angles. The second property inherent in conformal projections is that at any particular point, scale distortion is the same in any direction. This point is illustrated in the example by the fact that the meridian and parallel distortions are equal to each other at all intersections.

The following five pages illustrate a Stereographic Projection.

PERSPECTIVE MAP PROJECTIONS

EXAMPLE 2 - STEREOGRAPHIC PROJECTION

PROJECTION PARAMETERS -

CENTRAL LATITUDE = 40 DEGREES

CENTRAL LONGITUDE = 90 DEGREES

HEIGHT OF PROJECTION CENTER = -12742448.000 METERS

FOCAL LENGTH (SCALE FACTOR) = -12742448.000 METERS

SPHERE RADIUS = 6371224.000 METERS

LONGITUDE.....	90	80	70	60	50
LAT. 50					
X	-0.00	1411737.18	2720590.64	3841895.50	4721009.34
Y	12742448.00	12568441.75	12066612.20	11292150.08	10321642.19
MER.	2.0000	1.9851	1.9423	1.8762	1.7934
PAR.	2.0000	1.9851	1.9423	1.8762	1.7934
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 60					
X	-0.00	938009.54	1820841.42	2600577.24	3241889.38
Y	10692183.42	10586682.43	10279432.18	9796365.59	9175267.35
MER.	1.7041	1.6957	1.6712	1.6327	1.5832
PAR.	1.7041	1.6957	1.6712	1.6327	1.5832
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 70					
X	-0.00	562250.11	1097776.10	1582348.69	1996255.98
Y	8922358.14	8864353.79	8694107.61	8422399.70	8065593.86
MER.	1.4903	1.4859	1.4729	1.4523	1.4252
PAR.	1.4903	1.4859	1.4729	1.4523	1.4252
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 80					
X	-0.00	255809.90	501842.04	728907.43	928925.75
Y	7356855.78	7332571.55	7260840.36	7144931.85	6989994.35
MER.	1.3333	1.3315	1.3262	1.3177	1.3062
PAR.	1.3333	1.3315	1.3262	1.3177	1.3062
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 90					
X	-0.00	-0.00	-0.00	-0.00	-0.00
Y	5941901.09	5941901.09	5941901.09	5941901.09	5941901.09
MER.	1.1111	1.1111	1.1111	1.1111	1.1111
PAR.	1.1111	1.1111	1.1111	1.1111	1.1111
ANG.	1.1111	1.1111	1.1111	1.1111	1.1111
LAT. 80					
X	0.00	217815.59	430477.47	632843.48	819804.60
Y	4637871.78	4655434.25	4707826.02	4794148.25	4912864.40
MER.	1.1325	1.1338	1.1376	1.1440	1.1528
PAR.	1.1325	1.1338	1.1376	1.1440	1.1528
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 70					
X	0.00	406428.93	805622.82	1190157.15	1552244.72
Y	3414328.65	3444483.50	3534796.53	3684772.75	3893451.41
MER.	1.0718	1.0741	1.0809	1.0923	1.1082
PAR.	1.0718	1.0741	1.0809	1.0923	1.1082
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 60					
X	0.00	572090.86	1136958.30	1686957.30	2213600.73
Y	2246837.38	2285770.51	2402780.36	2598445.63	2873548.88
MER.	1.0311	1.0342	1.0435	1.0591	1.0810
PAR.	1.0311	1.0342	1.0435	1.0591	1.0810
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 50					
X	0.00	719303.37	1432848.83	2134283.11	2816031.50
Y	1114819.75	1159488.58	1294152.62	1520744.24	1842338.43
MER.	1.0077	1.0115	1.0230	1.0423	1.0697
PAR.	1.0077	1.0115	1.0230	1.0423	1.0697
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000

LONGITUDE.....	90	80	70	60	50
LAT. 40					
X	0.00	851309.16	1699347.24	2540174.00	3368443.47
Y	0.00	47874.76	192605.37	437505.40	788066.07
MER.	1.0000	1.0045	1.0180	1.0409	1.0737
PAR.	1.0000	1.0045	1.0180	1.0409	1.0737
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 30					
X	0.00	970389.81	1940709.20	2910260.49	3876994.93
Y	-1114819.75	-1065938.18	-917792.49	-665835.20	-302348.17
MER.	1.0077	1.0128	1.0284	1.0549	1.0931
PAR.	1.0077	1.0128	1.0284	1.0549	1.0931
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 20					
X	0.00	1078031.63	2159671.88	3248060.38	4345290.57
Y	-2246837.38	-2198952.12	-2053495.86	-1804966.80	-1443860.14
MER.	1.0311	1.0369	1.0547	1.0850	1.1291
PAR.	1.0311	1.0369	1.0547	1.0850	1.1291
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 10					
X	0.00	1174985.94	2357536.27	3555010.75	4774264.56
Y	-3414328.65	-3369351.81	-3232450.04	-2997557.31	-2654042.94
MER.	1.0718	1.0784	1.0986	1.1332	1.1838
PAR.	1.0718	1.0784	1.0986	1.1332	1.1838
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 0					
X	0.00	1261225.87	2534048.46	3830209.56	5161686.00
Y	-4637871.78	-4597710.23	-4475242.20	-4264328.57	-3954080.88
MER.	1.1325	1.1400	1.1629	1.2023	1.2604
PAR.	1.1325	1.1400	1.1629	1.2023	1.2604
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. -10					
X	0.00	1335776.11	2687026.06	4069770.02	5501130.51
Y	-5941901.09	-5908527.26	-5806596.92	-5630483.71	-5370108.74
MER.	1.2174	1.2260	1.2521	1.2973	1.3640
PAR.	1.2174	1.2260	1.2521	1.2973	1.3640
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. -20					
X	0.00	1396354.14	2811600.79	4265579.63	5780112.95
Y	-7356855.78	-7332360.25	-7257449.91	-7127679.27	-6935021.58
MER.	1.3333	1.3431	1.3731	1.4249	1.5020
PAR.	1.3333	1.3431	1.3731	1.4249	1.5020
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. -30					
X	0.00	1438694.61	2898814.36	4403054.66	5976810.99
Y	-8922358.14	-8908965.95	-8867974.21	-8796830.89	-8690902.98
MER.	1.4903	1.5016	1.5361	1.5960	1.6852
PAR.	1.4903	1.5016	1.5361	1.5960	1.6852
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. -40					
X	0.00	1455293.82	2933037.90	4457089.51	6054311.63
Y	-10692183.42	-10692183.42	-10692183.42	-10692183.42	-10692183.42
MER.	1.7041	1.7171	1.7571	1.8264	1.9298
PAR.	1.7041	1.7171	1.7571	1.8264	1.9298
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000

LONGITUDE.....	40	30	20	10	0
LAT. 50					
X	5335886.89	5691969.65	5813275.12	5733356.59	5488251.42
Y	9237041.51	8112659.28	7007798.45	5964757.89	5010423.49
MER.	1.7008	1.6049	1.5106	1.4216	1.3401
PAR.	1.7008	1.6049	1.5106	1.4216	1.3401
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 60					
X	3724348.11	4041756.60	4199426.25	4210577.24	4092853.57
Y	8459535.16	7692281.59	6911988.16	6150125.56	5430512.29
MER.	1.5262	1.4650	1.4029	1.3421	1.2848
PAR.	1.5262	1.4650	1.4029	1.3421	1.2848
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 70					
X	2325529.35	2562277.20	2704245.76	2753859.79	2717027.44
Y	7643643.52	7177963.50	6689545.93	6197558.26	5718499.97
MER.	1.3931	1.3578	1.3206	1.2833	1.2469
PAR.	1.3931	1.3578	1.3206	1.2833	1.2469
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 80					
X	1095321.43	1223264.16	1309749.03	1353534.76	1354974.14
Y	6802652.21	6590527.58	6361748.16	6124494.27	5886622.70
MER.	1.2924	1.2767	1.2598	1.2423	1.2247
PAR.	1.2924	1.2767	1.2598	1.2423	1.2247
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 90					
X	-0.00	-0.00	-0.00	-0.00	-0.00
Y	5941901.09	5941901.09	5941901.09	5941901.09	5941901.09
MER.	1.1111	1.1111	1.1111	1.1111	1.1111
PAR.	1.1111	1.1111	1.1111	1.1111	1.1111
ANG.	1.1111	1.1111	1.1111	1.1111	1.1111
LAT. 80					
X	986326.36	1127519.46	1238747.48	1315777.39	1354974.14
Y	5061746.98	5237812.36	5437253.81	5655386.65	5886622.70
MER.	1.1638	1.1768	1.1915	1.2076	1.2247
PAR.	1.1638	1.1768	1.1915	1.2076	1.2247
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 70					
X	1883598.12	2175352.68	2418082.29	2601948.39	2717027.44
Y	4159200.46	4479427.20	4850210.15	5265868.01	5718499.97
MER.	1.1284	1.1527	1.1809	1.2125	1.2469
PAR.	1.1284	1.1527	1.1809	1.2125	1.2469
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 60					
X	2707147.17	3156217.94	3547486.24	3865513.96	4092853.57
Y	3228781.84	3664291.02	4179028.18	4769872.43	5430512.29
MER.	1.1093	1.1440	1.1851	1.2321	1.2848
PAR.	1.1093	1.1440	1.1851	1.2321	1.2848
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 50					
X	3468615.63	4079903.31	4634305.76	5111988.83	5488251.42
Y	2262892.74	2786795.67	3418128.94	4159517.61	5010423.49
MER.	1.1056	1.1503	1.2042	1.2675	1.3401
PAR.	1.1056	1.1503	1.2042	1.2675	1.3401
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000

LONGITUDE.....	40	30	20	10	0
LAT. 40					
X	4176531.76	4953461.50	5683567.09	6344883.24	6907336.45
Y	1251860.20	1838296.93	2558086.76	3422194.28	4439950.29
MER.	1.1171	1.1719	1.2392	1.3201	1.4153
PAR.	1.1171	1.1719	1.2392	1.3201	1.4153
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 30					
X	4836584.26	5781157.52	6697545.62	7564858.05	8351245.20
Y	183729.37	806949.55	1585340.49	2539961.51	3693555.04
MER.	1.1443	1.2098	1.2917	1.3922	1.5136
PAR.	1.1443	1.2098	1.2917	1.3922	1.5136
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 20					
X	5451593.77	6564106.26	7674962.10	8768342.18	9815977.84
Y	-956169.27	-322711.33	481647.59	1488669.71	2736865.18
MER.	1.1887	1.2660	1.3642	1.4872	1.6396
PAR.	1.1887	1.2660	1.3642	1.4872	1.6396
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 10					
X	6021129.02	7299483.63	8609607.63	9945288.24	11288816.35
Y	-2185885.79	-1570434.47	-776694.10	236869.36	1524829.01
MER.	1.2527	1.3433	1.4602	1.6095	1.7992
PAR.	1.2527	1.3433	1.4602	1.6095	1.7992
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 0					
X	6540643.34	7979108.00	9488081.93	11075565.28	12742448.00
Y	-3527780.01	-2961155.49	-2219789.49	-1255313.41	0.00
MER.	1.3401	1.4461	1.5848	1.7652	2.0000
PAR.	1.3401	1.4461	1.5848	1.7652	2.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. -10					
X	6999895.05	8587047.45	10286146.57	12123238.50	14125538.62
Y	-5009753.68	-4526095.07	-3885063.45	-3036857.19	-1907997.29
MER.	1.4563	1.5803	1.7446	1.9620	2.2513
PAR.	1.4563	1.5803	1.7446	1.9620	2.2513
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. -20					
X	7380227.52	9095680.42	10962929.84	13027702.11	15348237.81
Y	-6666803.66	-6303890.23	-5817663.01	-5164954.72	-4279355.38
MER.	1.6092	1.7543	1.9486	2.2096	2.5636
PAR.	1.6092	1.7543	1.9486	2.2096	2.5636
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. -30					
X	7649962.18	9459280.84	11451897.29	13690526.04	16261690.14
Y	-8542813.10	-8341287.44	-8069187.87	-7700093.77	-7192154.71
MER.	1.8099	1.9796	2.2087	2.5195	2.9472
PAR.	1.8099	1.9796	2.2087	2.5195	2.9472
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. -40					
X	7756600.99	9603693.17	11647311.36	13957654.17	16634084.50
Y	-10692183.42	-10692183.42	-10692183.42	-10692183.42	-10692183.42
MER.	2.0746	2.2721	2.5396	2.9039	3.4082
PAR.	2.0746	2.2721	2.5396	2.9039	3.4082
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000

6.5 Example 3 - The Orthographic Projection

Rather than illustrating an Orthographic Projection with the projection center located at an arbitrary latitude of 40 degrees as was done in Examples 1 and 2, a Polar Orthographic is illustrated in Example 3. A Polar Orthographic is a perspective projection in which the projection axis is coincident with the axis of rotation of the datum body, and the projection center is located at an infinite distance along this axis. In this type of projection the most prominent properties are that all parallels of latitude are projected in true size as concentric circles with centers at the pole and all meridians are projected as straight lines radiating from the pole. These properties are illustrated in the following example by the parallel distortions all being equal to unity, or, in other words, the distances along the parallels measured on the projection plane are equal to the distances along the parallels measured on the datum surface. Also, it is noted that all the angular distortions are equal to zero. This is due to the fact that the parallels are all concentric circles and the meridians are all straight lines radiating from the center of the concentric circles. Thus, all intersections of meridians and parallels form right angles.

The following five pages illustrate an Orthographic Projection.

PERSPECTIVE MAP PROJECTIONS

EXAMPLE 3 - ORTHOGRAPHIC PROJECTION

PROJECTION PARAMETERS -

CENTRAL LATITUDE = 90 DEGREES

CENTRAL LONGITUDE = 90 DEGREES

HEIGHT OF PROJECTION CENTER = INFINITY

FOCAL LENGTH = INFINITY

SPHERE RADIUS = 6371224.000 METERS

LONGITUDE.....		90	80	70	60	50
LAT.	0					
X		-0.00	1106351.44	2179086.95	3185612.00	4095343.85
Y		6371224.00	6274430.79	5986992.18	5517641.84	4880640.74
MER.		0.0000	0.0000	0.0000	0.0000	0.0000
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	10					
X		-0.00	1089543.47	2145981.72	3137215.40	4033126.37
Y		6274430.79	6179108.09	5896036.31	5433816.46	4806492.84
MER.		0.1736	0.1736	0.1736	0.1736	0.1736
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	20					
X		-0.00	1039630.28	2047671.92	2993496.09	3848364.39
Y		5986992.18	5896036.31	5625932.37	5184887.32	4586302.09
MER.		0.3420	0.3420	0.3420	0.3420	0.3420
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	30					
X		-0.00	958128.45	1887144.65	2758820.92	3546671.81
Y		5517641.84	5433816.46	5184887.32	4778418.00	4226758.87
MER.		0.5000	0.5000	0.5000	0.5000	0.5000
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	40					
X		-0.00	847514.37	1669277.45	2440320.37	3137215.40
Y		4880640.74	4806492.84	4586302.09	4226758.87	3738787.72
MER.		0.6428	0.6428	0.6428	0.6428	0.6428
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	50					
X		-0.00	711149.00	1400690.09	2047671.92	2632436.28
Y		4095343.85	4033126.37	3848364.39	3546671.81	3137215.40
MER.		0.7660	0.7660	0.7660	0.7660	0.7660
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	60					
X		-0.00	553175.72	1089543.47	1592806.00	2047671.92
Y		3185612.00	3137215.40	2993496.09	2758820.92	2440320.37
MER.		0.8660	0.8660	0.8660	0.8660	0.8660
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	70					
X		-0.00	378394.48	745291.63	1089543.47	1400690.09
Y		2179086.95	2145981.72	2047671.92	1887144.65	1669277.45
MER.		0.9397	0.9397	0.9397	0.9397	0.9397
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	80					
X		-0.00	192115.91	378394.48	553175.72	711149.00
Y		1106351.44	1089543.47	1039630.28	958128.45	847514.37
MER.		0.9848	0.9848	0.9848	0.9848	0.9848
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000

LONGITUDE.....		90	80	70	60	50
LAT.	90					
X		-0.00	-0.00	-0.00	-0.00	-0.00
Y		-0.00	-0.00	-0.00	-0.00	-0.00
MER.		1.1111	1.1111	1.1111	1.1111	1.1111
PAR.		1.1111	1.1111	1.1111	1.1111	1.1111
ANG.		1.1111	1.1111	1.1111	1.1111	1.1111
LAT.	80					
X		0.00	192115.91	378394.48	553175.72	711149.00
Y		-1106351.44	-1089543.47	-1039630.28	-958128.45	-847514.37
MER.		0.9848	0.9848	0.9848	0.9848	0.9848
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	70					
X		0.00	378394.48	745291.63	1089543.47	1400690.09
Y		-2179086.95	-2145981.72	-2047671.92	-1887144.65	-1669277.45
MER.		0.9397	0.9397	0.9397	0.9397	0.9397
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	60					
X		0.00	553175.72	1089543.47	1592806.00	2047671.92
Y		-3185612.00	-3137215.40	-2993496.09	-2758820.92	-2440320.37
MER.		0.8660	0.8660	0.8660	0.8660	0.8660
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	50					
X		0.00	711149.00	1400690.09	2047671.92	2632436.28
Y		-4095343.85	-4033126.37	-3848364.39	-3546671.81	-3137215.40
MER.		0.7660	0.7660	0.7660	0.7660	0.7660
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	40					
X		0.00	847514.37	1669277.45	2440320.37	3137215.40
Y		-4880640.74	-4806492.84	-4586302.09	-4226758.87	-3738787.72
MER.		0.6428	0.6428	0.6428	0.6428	0.6428
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	30					
X		0.00	958128.45	1887144.65	2758820.92	3546671.81
Y		-5517641.84	-5433816.46	-5184887.32	-4778418.00	-4226758.87
MER.		0.5000	0.5000	0.5000	0.5000	0.5000
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	20					
X		0.00	1039630.28	2047671.92	2993496.09	3848364.39
Y		-5986992.18	-5896036.31	-5625932.37	-5184887.32	-4586302.09
MER.		0.3420	0.3420	0.3420	0.3420	0.3420
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	10					
X		0.00	1089543.47	2145981.72	3137215.40	4033126.37
Y		-6274430.79	-6179108.09	-5896036.31	-5433816.46	-4806492.84
MER.		0.1736	0.1736	0.1736	0.1736	0.1736
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000

LONGITUDE.....		40	30	20	10	0
LAT.	0					
	X	4880640.74	5517641.84	5986992.18	6274430.79	6371224.00
	Y	4095343.85	3185612.00	2179086.95	1106351.44	0.00
MER.		0.0000	0.0000	0.0000	0.0000	0.0000
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	10					
	X	4806492.84	5433816.46	5896036.31	6179108.09	6274430.79
	Y	4033126.37	3137215.40	2145981.72	1089543.47	0.00
MER.		0.1736	0.1736	0.1736	0.1736	0.1736
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	20					
	X	4586302.09	518[illegible]7.32	5625932.37	5896036.31	5986992.18
	Y	3848364.39	299[illegible]6.09	2047671.92	1039630.28	0.00
MER.		0.3420	0.3420	0.3420	0.3420	0.3420
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	30					
	X	4226758.87	4778418.00	5184887.32	5433816.46	5517641.84
	Y	3546671.81	2758820.92	1887144.65	958128.45	0.00
MER.		0.5000	0.5000	0.5000	0.5000	0.5000
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	40					
	X	3738787.72	4226758.87	4586302.09	4806492.84	4880640.74
	Y	3137215.40	2440320.37	1669277.45	847514.37	0.00
MER.		0.6428	0.6428	0.6428	0.6428	0.6428
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	50					
	X	3137215.40	3546671.81	3848364.39	4033126.37	4095343.85
	Y	2632436.28	2047671.92	1400690.09	711149.00	-0.00
MER.		0.7660	0.7660	0.7660	0.7660	0.7660
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	60					
	X	2440320.37	2758820.92	2993496.09	3137215.40	3185612.00
	Y	2047671.92	1592806.00	1089543.47	553175.72	-0.00
MER.		0.8660	0.8660	0.8660	0.8660	0.8660
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	70					
	X	1669277.45	1887144.65	2047671.92	2145981.72	2179086.95
	Y	1400690.09	1089543.47	745291.63	378394.48	-0.00
MER.		0.9397	0.9397	0.9397	0.9397	0.9397
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000
LAT.	80					
	X	847514.37	958128.45	1039630.28	1089543.47	1106351.44
	Y	711149.00	553175.72	378394.48	192115.91	-0.00
MER.		0.9848	0.9848	0.9848	0.9848	0.9848
PAR.		1.0000	1.0000	1.0000	1.0000	1.0000
ANG.		0.0000	0.0000	0.0000	0.0000	0.0000

LONGITUDE.....	40	30	20	10	0
LAT. 90					
X	-0.00	-0.00	-0.00	-0.00	-0.00
Y	-0.00	-0.00	-0.00	-0.00	-0.00
MER.	1.1111	1.1111	1.1111	1.1111	1.1111
PAR.	1.1111	1.1111	1.1111	1.1111	1.1111
ANG.	1.1111	1.1111	1.1111	1.1111	1.1111
LAT. 80					
X	847514.37	958128.45	1039630.28	1089543.47	1106[illegible]51.44
Y	-711149.00	-553175.72	-378394.48	-192115.91	-0.00
MER.	0.9848	0.9848	0.9848	0.9848	0.9848
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 70					
X	1669277.45	1887144.65	2047671.92	2145981.72	2179086.95
Y	-1400690.09	-1089543.47	-745291.63	-378394.48	-0.00
MER.	0.9397	0.9397	0.9397	0.9397	0.9397
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 60					
X	2440320.37	2758820.92	2993496.09	3137215.40	3185612.00
Y	-2047671.92	-1592806.00	-1089543.47	-553175.72	-0.00
MER.	0.8660	0.8660	0.8660	0.8660	0.8660
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 50					
X	3137215.40	3546671.81	3848364.39	4033126.37	4095343.85
Y	-2632436.28	-2047671.92	-1400690.09	-711149.00	-0.00
MER.	0.7660	0.7660	0.7660	0.7660	0.7660
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 40					
X	3738787.72	4226758.87	4586302.09	4806492.84	4880640.74
Y	-3137215.40	-2440320.37	-1669277.45	-847514.37	0.00
MER.	0.6428	0.6428	0.6428	0.6428	0.6428
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 30					
X	4226758.87	4778418.00	5184887.32	5433816.46	5517641.84
Y	-3546671.81	-2758820.92	-1887144.65	-958128.45	0.00
MER.	0.5000	0.5000	0.5000	0.5000	0.5000
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 20					
X	4586302.09	5184887.32	5625932.37	5896036.31	5986992.18
Y	-3848364.39	-2993496.09	-2047671.92	-1039630.28	0.00
MER.	0.3420	0.3420	0.3420	0.3420	0.3420
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 10					
X	4806492.84	5433816.46	4896036.31	6179108.09	6274430.79
Y	-4033126.37	-3137215.40	-2145981.72	-1089543.47	0.00
MER.	0.1736	0.1736	0.1736	0.1736	0.1736
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000

6.6 Example 4 - A General Projection Of A Spherical Datum

In Examples 1, 2, and 3, the parameter h, the height of the projection center above the datum shpere, was set equal to a particular value in relationship to the datum sphere in order that the resulting projection would exhibit certain desired properties. In Example 4, a completely arbitrary value of approximately 700 miles has been selected for the parameter h. Thus, ideally, Example 4 illustrates the type of projection that would be obtained by taking a vertical photograph of the datum sphere from a height of approximately 700 miles.

In the following example the computer was programmed to print out the coordinates of only those meridian-parallel intersections that are visible from the projection center. By visible, the author means that the intersection point is not beyond the horizon as viewed from the projection center. The relationship between latitude and longitude values on the horizon is calculated by setting $LP_{F'}$ equal to 0 in equation (23), which results in the following equation:

$$\cos \theta = \cos \phi_c \cos \phi \cos \lambda + \sin \phi_c \sin \phi \tag{87}$$

It is interesting to note that the shape of the projected grid in the following example actually resembles the shape of the grid that would be visible from the projection center.

The following two pages illustrate a general projection of a spherical datum.

PERSPECTIVE MAP PROJECTIONS

EXAMPLE 4 - GENERAL PROJECTION

PROJECTION PARAMETERS -

CENTRAL LATITUDE = 40 DEGREES

CENTRAL LONGITUDE = 90 DEGREES

HEIGHT OF PROJECTION CENTER = 1126542.900 METERS

FOCAL LENGTH (SCALE FACTOR) = 1126542.900 METERS

SPHERE RADIUS = 6371224.000 METERS

LONGITUDE.....	90	80	70	60	50
LAT. 70					
X	0.00	212555.86	403501.10		
Y	1812374.71	1801410.01	1770424.38		
MER.	0.0351	0.1495	0.2695		
PAR.	0.5689	0.5421	0.4696		
ANG.	0.0000	79.6911	89.7728		
LAT. 60					
X	0.00	402607.95	740325.02	976410.19	1108127.44
Y	1624884.83	1608607.03	1564559.07	1503979.25	1438497.17
MER.	0.3328	0.3861	0.4715	0.5142	0.5060
PAR.	0.7457	0.6845	0.5306	0.3475	0.1920
ANG.	0.0000	35.2465	56.0944	70.6413	89.7929
LAT. 50					
X	0.00	630323.72	1117097.50	1403461.63	1515120.17
Y	1018814.33	1016056.92	1008965.24	1000010.81	991240.37
MER.	0.7623	0.7612	0.7354	0.6673	0.5740
PAR.	0.9209	0.8069	0.5424	0.2668	0.0608
ANG.	0.0000	15.7277	26.8938	33.7398	43.8255
LAT. 40					
X	0.00	806833.61	1390891.78	1689227.46	1765997.41
Y	0.00	45373.60	157644.78	290943.11	413164.91
MER.	1.0000	0.9517	0.8301	0.6833	0.5482
PAR.	1.0000	0.8529	0.5357	0.2539	0.1310
ANG.	0.0000	0.7103	5.9601	24.3073	74.7379
LAT. 30					
X	0.00	838315.20	1438161.72	1736647.90	
Y	-1018814.33	-920859.00	-680129.73	-397325.71	
MER.	0.7623	0.7302	0.6481	0.5463	
PAR.	0.9209	0.7987	0.5438	0.3357	
ANG.	0.0000	17.1789	37.3981	65.2857	
LAT. 20					
X	0.00	741046.79	1290611.26		
Y	-1624884.83	-1511575.70	-1227160.90		
MER.	0.3328	0.3572	0.3854		
PAR.	0.7457	0.6722	0.5137		
ANG.	0.0000	37.5277	66.9396		
LAT. 10					
X	0.00	597822.64			
Y	-1812374.71	-1714296.95			
MER.	0.0351	0.1408			
PAR.	0.5689	0.5304			
ANG.	0.0000	85.9367			

6.7 Example 5 - A general Projection Of An Ellipsoidal Datum

Example 5 illustrates a projection analogous to Example 4 with the exception that an ellipsoidal datum surface is utilized rather than a spherical datum surface. The same value of h, approximately 700 miles, is also used in Example 5. Therefore, Example 5 illustrates to a better approximation than Example 4 the projection of the actual meridians and parallels on the surface of the earth.

Equation (87) was again utilized in obtaining those intersections of meridians and parallels that are visible from the projection center. While equation (87) is an exact formula when dealing with a spherical datum surface, this equation is only an approximation when dealing with an ellipsoidal datum surface. The author wishes to emphasize, however, that the calculated values are exact and the approximation only involves the determination as to which intersections are visible.

The following two pages illustrate a general projection of an ellipsoidal datum.

PERSPECTIVE MAP PROJECTIONS

EXAMPLE 5 - GENERAL PROJECTION

PROJECTION PARAMETERS -

CENTRAL LATITUDE = 40 DEGREES

CENTRAL LONGITUDE = 90 DEGREES

HEIGHT OF PROJECTION CENTER = 1126542.900 METERS

FOCAL LENGTH (SCALE FACTOR) = 1126542.900 METERS

ELLIPSOID MAJOR SEMI DIAMETER = 6378388.000 METERS

ELLIPSOID FLATTENING = 1/297.0

LONGITUDE.....	90	80	70	60	50
LAT. 70					
X	0.00	213247.66	404766.70		
Y	1812769.58	1801764.18	1770668.50		
MER.	0.0352	0.1507	0.2716		
PAR.	0.5685	0.5416	0.4690		
ANG.	0.0000	79.7439	89.8051		
LAT. 60					
X	0.00	403959.06	742643.99	979168.62	1110890.16
Y	1624973.58	1608639.02	1564450.96	1503710.60	1438096.83
MER.	0.3344	0.3881	0.4740	0.5167	0.5082
PAR.	0.7455	0.6842	0.5300	0.3466	0.1912
ANG.	0.0000	35.2851	56.1342	70.6878	89.9032
LAT. 50					
X	0.00	632220.67	1120121.61	1406725.49	1518077.95
Y	1018253.01	1015508.42	1008452.65	999549.64	990836.13
MER.	0.7637	0.7626	0.7366	0.6680	0.5744
PAR.	0.9209	0.8067	0.5416	0.2658	0.0600
ANG.	0.0000	15.7402	26.9041	33.7372	43.8693
LAT. 40					
X	0.00	808767.25	1393808.42	1692169.17	1768497.56
Y	-0.00	45482.34	157975.36	291449.78	413749.83
MER.	1.0000	0.9516	0.8297	0.6827	0.5475
PAR.	1.0000	0.8525	0.5349	0.2531	0.1307
ANG.	0.0000	0.7120	5.9752	24.3909	74.4825
LAT. 30					
X	0.00	840043.44	1440743.42	1739222.20	
Y	-1016690.79	-918639.64	-677765.04	-394953.84	
MER.	0.7627	0.7306	0.6483	0.5463	
PAR.	0.9211	0.7985	0.5431	0.3348	
ANG.	0.0000	17.1835	37.4138	65.3518	
LAT. 20					
X	0.00	742761.38	1293251.00		
Y	-1621426.01	-1508007.55	-1223424.93		
MER.	0.3339	0.3584	0.3865		
PAR.	0.7463	0.6725	0.5133		
ANG.	0.0000	37.5217	66.9340		
LAT. 10					
X	0.00	599603.54			
Y	-1809173.44	-1710917.70			
MER.	0.0359	0.1418			
PAR.	0.5700	0.5311			
ANG.	0.0000	85.6795			

6.8 Example 6 - An Orthographic Projection Of An Ellipsoidal Datum

Example 6 is included as a final illustration of the utility of the mapping equations derived previously in this paper. The Orthographic Projection is a rather commonly used projection, but to the best of the author's knowledge a spherical datum surface is always used for this projection. Example 6 illustrates a Polar Orthographic Projection using an ellipsoidal datum surface. This type of projection more closely approximates the projection of the actual meridians and parallels of the earth than a projection using a spherical datum surface. Therefore, in this respect, it appears to be a superior type projection.

It can be seen that the properties noted in Example 3 are also evident in Example 6. As in Example 3, the linear distortions along the parallels and the angular distortions in the intersections of meridians and parallels are both equal to zero.

A disadvantage in using an ellipsoidal datum surface rather than a spherical datum surface is obviously the more complicated mapping equations required for the former. However, as the use of electronic computers increases, this disadvantage correspondingly decreases.

The following five pages illustrate an Orthographic Projection of an ellipsoidal datum.

PERSPECTIVE MAP PROJECTIONS

EXAMPLE 6 - ORTHOGRAPHIC PROJECTION

PROJECTION PARAMETERS -

CENTRAL LATITUDE = 90 DEGREES

CENTRAL LONGITUDE = 90 DEGREES

HEIGHT OF PROJECTION CENTER = INFINITY

FOCAL LENGTH = INFINITY

ELLIPSOID MAJOR SEMI DIAMETER = 6378388.000 METERS

ELLIPSOID FLATTENING = 1/297.0

LONGITUDE.....	90	80	70	60	50
LAT. 0					
X	-0.00	1107595.45	2181537.18	3189194.00	4099948.78
Y	6378388.00	6281485.95	5993724.14	5523846.04	4886128.68
MER.	0.0000	0.0000	0.0000	0.0000	0.0000
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 10					
X	-0.00	1090879.16	2148612.51	3141061.36	4038070.65
Y	6282122.72	6186683.16	5903264.36	5440477.87	4812385.20
MER.	0.1736	0.1736	0.1736	0.1736	0.1736
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 20					
X	-0.00	1041208.76	2050780.92	2998041.13	3854207.39
Y	5996082.27	5904988.31	5634474.26	5192759.57	4593265.50
MER.	0.3420	0.3420	0.3420	0.3420	0.3420
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 30					
X	-0.00	960012.87	1890856.23	2764246.89	3553647.30
Y	5528493.78	5444503.53	5195084.81	4787816.06	4235071.94
MER.	0.5000	0.5000	0.5000	0.5000	0.5000
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 40					
X	-0.00	849648.17	1673480.22	2446464.41	3145114.02
Y	4892928.82	4818594.24	4597849.11	4237400.66	3748203.93
MER.	0.6428	0.6428	0.6428	0.6428	0.6428
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 50					
X	-0.00	713357.13	1405039.26	2054029.99	2640610.06
Y	4108059.98	4045649.32	3860313.65	3557684.31	3146956.52
MER.	0.7674	0.7674	0.7674	0.7674	0.7674
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 60					
X	-0.00	555199.15	1093528.86	1598632.25	2055162.00
Y	3197264.49	3148690.86	3004445.85	2768912.27	2449246.70
MER.	0.8704	0.8704	0.8704	0.8704	0.8704
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 70					
X	-0.00	379949.37	748354.18	1094020.62	1406445.8[illegible]
Y	2188041.24	2154799.98	2056086.21	1894899.30	1676136.84
MER.	0.9470	0.9470	0.9470	0.9470	0.9470
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 80					
X	-0.00	192962.01	380060.97	555611.97	714280.96
Y	1111223.94	1094341.95	1044208.94	962348.16	851246.93
MER.	0.9942	0.9942	0.9942	0.9942	0.9942
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000

LONGITUDE.....	90	80	70	60	50
LAT. 90					
X	-0.00	-0.00	-0.00	-0.00	-0.00
Y	0.00	-0.00	-0.00	-0.00	-0.00
MER.	1.1111	1.1111	1.1111	1.1111	1.1111
PAR.	1.1111	1.1111	1.1111	1.1111	1.1111
ANG.	1.1111	1.1111	1.1111	1.1111	1.1111
LAT. 80					
X	0.00	192962.01	380060.97	555611.97	714280.98
Y	-1111223.94	-1094341.95	-1044208.94	-962348.16	-851246.93
MER.	0.9942	0.9942	0.9942	0.9942	0.9942
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 70					
X	0.00	379949.37	748354.18	1094020.62	1406445.83
Y	-2188041.24	-2154799.98	-2056086.21	-1894899.30	-1676136.84
MER.	0.9470	0.9470	0.9470	0.9470	0.9470
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 60					
X	0.00	555199.15	1093528.86	1598632.25	2055162.30
Y	-3197264.49	-3148690.86	-3004445.85	-2768912.27	-2449246.70
MER.	0.8704	0.8704	0.8704	0.8704	0.8704
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 50					
X	0.00	713357.13	1405039.26	2054029.99	2640610.06
Y	-4108059.98	-4045649.32	-3860313.65	-3557684.31	-3146956.52
MER.	0.7674	0.7674	0.7674	0.7674	0.7674
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 40					
X	0.00	849648.17	1673480.22	2446464.41	3145114.02
Y	-4892928.82	-4818594.24	-4597849.11	-4237400.66	-3748200.93
MER.	0.6428	0.6428	0.6428	0.6428	0.6428
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 30					
X	0.00	960012.87	1890856.23	2764246.89	3553647.30
Y	-5528493.78	-5444503.53	-5195084.81	-4787816.06	-4235071.94
MER.	0.5000	0.5000	0.5000	0.5000	0.5000
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 20					
X	0.00	1041208.76	2050780.92	2998041.13	3854207.39
Y	-5996082.27	-5904988.31	-5634474.26	-5192759.57	-4593265.50
MER.	0.3420	0.3420	0.3420	0.3420	0.3420
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 10					
X	0.00	1090879.16	2148612.51	3141061.36	4038070.65
Y	-6282122.72	-6186683.16	-5903264.36	-5440477.87	-4812385.20
MER.	0.1736	0.1736	0.1736	0.1736	0.1736
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000

LONGITUDE.....	40	30	20	10	0
LAT. 0					
X	4886128.68	5523846.04	5993724.14	6281485.95	6378388.00
Y	4099948.78	3189194.00	2181537.18	1107595.45	0.00
MER.	0.0000	0.0000	0.0000	0.0000	0.0000
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 10					
X	4812385.20	5440477.87	5903264.36	6186683.16	6282122.72
Y	4038070.65	3141061.36	2148612.51	1090879.16	0.00
MER.	0.1736	0.1736	0.1736	0.1736	0.1736
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 20					
X	4593265.50	5192759.57	5634474.26	5904988.31	5996082.27
Y	3854207.39	2998041.13	2050780.92	1041208.76	0.00
MER.	0.3420	0.3420	0.3420	0.3420	0.3420
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 30					
X	4235071.94	4787816.06	5195084.81	5444503.53	5528493.78
Y	3553647.30	2764246.89	1890856.23	960012.87	0.00
MER.	0.5000	0.5000	0.5000	0.5000	0.5000
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 40					
X	3748200.93	4237400.66	4597849.11	4818594.24	4892928.82
Y	3145114.02	2446464.41	1673480.22	849648.17	0.00
MER.	0.6428	0.6428	0.6428	0.6428	0.6428
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 50					
X	3146956.52	3557684.31	3860313.65	4045649.32	4108059.98
Y	2640610.06	2054029.99	1405039.26	713357.13	-0.00
MER.	0.7674	0.7674	0.7674	0.7674	0.7674
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 60					
X	2449246.70	2768912.27	3004445.85	3148690.86	3197264.49
Y	2055162.00	1598632.25	1093528.86	555199.15	-0.00
MER.	0.8704	0.8704	0.8704	0.8704	0.8704
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 70					
X	1676136.84	1894899.30	2056086.21	2154799.98	2188041.24
Y	1406445.80	1094020.62	748354.18	379949.37	-0.00
MER.	0.9470	0.9470	0.9470	0.9470	0.9470
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 80					
X	851246.93	962348.16	1044208.94	1094341.95	1111223.94
Y	714280.98	555611.97	380060.97	192962.01	-0.00
MER.	0.9942	0.9942	0.9942	0.9942	0.9942
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000

LONGITUDE.....	40	30	20	10	0
LAT. 90					
X	-0.00	-0.00	-0.00	-0.00	-0.00
Y	-0.00	-0.00	-0.00	-0.00	-0.00
MER.	1.1111	1.1111	1.1111	1.1111	1.1111
PAR.	1.1111	1.1111	1.1111	1.1111	1.1111
ANG.	1.1111	1.1111	1.1111	1.1111	1.1111
LAT. 80					
X	851246.93	962348.16	1044208.94	1094341.95	1111223.94
Y	-714280.98	-555611.97	-380060.97	-192962.01	-0.00
MER.	0.9942	0.9942	0.9942	0.9942	0.9942
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 70					
X	1676136.84	1894899.30	2056086.21	2154799.98	2188041.24
Y	-1406445.80	-1094020.62	-748354.18	-379949.37	-0.00
MER.	0.9470	0.9470	0.9470	0.9470	0.9470
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 60					
X	2449246.70	2768912.27	3004445.85	3148690.86	3197264.49
Y	-2055162.00	-1598632.25	-1093528.86	-555199.15	-0.00
MER.	0.8704	0.8704	0.8704	0.8704	0.8704
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 50					
X	3146956.52	3557684.31	3860313.65	4045649.32	4108059.98
Y	-2640610.06	-2054029.99	-1405039.26	-713357.13	-0.00
MER.	0.7674	0.7674	0.7674	0.7674	0.7674
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 40					
X	3748200.93	4237400.66	4597849.11	4818594.24	4892928.82
Y	-3145114.02	-2446464.41	-1673480.22	-849648.17	0.00
MER.	0.6428	0.6428	0.6428	0.6428	0.6428
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 30					
X	4235071.94	4787816.06	5195084.81	5444503.53	5528493.78
Y	-3553647.30	-2764246.89	-1890856.23	-960012.87	0.00
MER.	0.5000	0.5000	0.5000	0.5000	0.5000
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 20					
X	4593265.50	5192759.57	5634474.26	5904988.31	5996082.27
Y	-3854207.39	-2998041.13	-2050780.92	-1041208.76	0.00
MER.	0.3420	0.3420	0.3420	0.3420	0.3420
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000
LAT. 10					
X	4812385.20	5440477.87	5903264.36	6186683.16	6282122.72
Y	-4038070.65	-3141061.36	-2148612.51	-1090879.16	0.00
MER.	0.1736	0.1736	0.1736	0.1736	0.1736
PAR.	1.0000	1.0000	1.0000	1.0000	1.0000
ANG.	0.0000	0.0000	0.0000	0.0000	0.0000

7. SUMMARY AND CONCLUSIONS

The primary goal of this paper, as stated in the Introduction, was to develop general mapping equations for the perspective projection which would be suitable both for spherical and ellipsoidal datum surfaces. A careful method of proceeding from the known to the unknown was utilized in developing the desired equations. First, mapping equations for selected perspective projections of a spherical datum surface were derived utilizing already validated methods. Next, mapping equations for a spherical datum surface were derived by a novel photogrammetric method. These mapping equations were proved equal to those derived earlier. After having at least partially proved the photogrammetric method to be valid by its use on a spherical datum surface, this method was used to derive mapping equations for an ellipsoidal datum surface. Equations (78) and (81), the General Perspective Projection Mapping Equations, were the results of the derivation. Equations (78) and (81) were proved to be general equations encompassing all the special cases that had been discussed prior to their derivation. It is evident from the derivation of these equations that they could be made to be even more general by providing for the possibility that the arbitrary point P that is to be projected is not on the datum surface but at some distance above or below it. The alteration to equations (78) and (81) to take into consideration this distance would be accomplished by first changing the coordinates of point

P, as given by equations (62), (63), and (64), to account for the distance and then by using these new coordinates in the remainder of the derivation. Finally, as a means of vizualizing certain perspective projections, an empirical method was developed to compute various distortions in the projections and examples of several projected grids were illustrated.

The author believes that equations (78) and (81) derived in this thesis are useful equations in that they are truly General Perspective Projection Mapping Equations suitable for both spherical and ellipsoidal datum surfaces that can be used regardless of the location of the projection center.

APPENDIX

The computer programs utilized to produce all examples included in this thesis were basically very similar. The actual equations used to calculate the desired data were identical in all the examples, and the primary differences in the programs concerned format. Thus, although this appendix is devoted to an explanation of the computer program used to produce Example 5, it essentially explains the computer programs used for all the included examples. The IBM 7094 computer of the Ohio State University was employed to produce the examples; and the programming language used was SCATRAN, a program language taught and used at Ohio State University.

A complete and detailed explanation of the program will not be given, but sufficient explanation will be given to allow the reader to understand the general principles of the program. A more thorough discussion of the computer program is not being presented due to the authors belief that the program is not an end product but only a tool that was used to illustrate and verify the concepts developed in this thesis.

In succeeding paragraphs, first the major terms used in the program will be identified, and then the functions of the Source Language Statements will be explained. The program itself is included at the end of this appendix.

The following is a list of the major terms that were

components of the program used to produce Example 5 of this thesis:

X - the abscissa or X coordinate of a particular point in meters

Y - the ordinate or Y coordinate of a particular point in meters

V - a longitude value used in printing out the grid format

W - a latitude value used in printing out the grid format

IMP - an angular distortion in the intersection of a meridian and a parallel

RDM - a ratio of a meridian distance on the projection plane to a meridian distance on a datum surface

RDP - a ratio of a parallel distance on the projection plane to a parallel distance on a datum surface

RAD - a conversion factor to convert degrees to radians

A - the sphere radius or ellipsoid semi-major axis in meters

FLAT - the ellipsoid flattening

HC - the height of the projection center above the datum surface in meters

F - the distance between the projection center and the projection plane in meters

BC - the central latitude in degrees

BP - the latitude of a particular point in degrees

LC - the central longitude in degrees

LP - the longitude of a particular point in degrees

DEGCH - a small number of degrees used to locate a point very close to a primary point. In the following definitions the

primary point, which is the intersection of a particular meridian and a particular parallel, will be identified as P_0. Figure 16 illustrates P_0 and its relationship to P_1 and P_2.

BPRB - the latitude of P_1 in radians

LPRL - the longitude of P_2 in radians

U - a term used to indicate the computation of an Orthographic Projection. When an Orthographic Projection is being calculated, U is set equal to 0 and F is set equal to 1.

E2 - the square of the eccentricity of an ellipsoid

A1, B1, C1, D1, E1, F1 - terms used in equation (83)

NP - the radius of curvature in the prime vertical at a particular point

NC - the radius of curvature in the prime veritcal along the projection axis

NPB - the radius of curvature in the prime vertical at P_1

DENOM - the denominator of equations (78) and (81)

XB - X coordinate of P_1

YB - Y coordinate of P_1

DENO - the denominator of equations (78) and (81) used to calculate XB and YB

XL - X coordinate of P_2

YL - Y coordinate of P_2

DEN - the denominator of equations (78) and (81) used to calculate XL and YL

DXB - the absolute difference in the X coordinates of P_0 and P_1

DYB - the absolute difference in the Y coordinates of P_0 and P_1

AB - the arctan of $\frac{DYB}{DXB}$

DXL - the absolute difference in the X coordinates of P_0 and P_2

DYL - the absolute difference in the Y coordinates of P_0 and P_2

AL - the arctan of $\frac{DYL}{DXL}$

DMM - the distance on the projection plane between P_0 and P_1

DPM - the distance on the projection plane between P_0 and P_2

DME - distance on the datum surface between P_0 and P_1

DPE - the distance on the datum surface between P_0 and P_2

The following is a brief explanation of the Source Language Statements of the program included at the end of this appendix:

1 - 35 : set up the projection parameters and are self explanatory when considered with the term definitions given above

36 - 45 : calculate the X and Y coordinates of P_0

46 - 71 : check the location of P_0 to determine if it is visible from the projection center

72 - 80 : calculate the X and Y coordinates of P_1

81 - 87 : calculate the X and Y coordinates of P_2

88 - 105 : calculate the angular distortion in the projected intersection of a parallel and a meridian

106 - 116 : calculate the meridian and parallel linear distortions

117 - 153 : dictate the size and shape of grid to be computed

154 - 159 : calculate the longitude values that will be used for column headings in the grid

160 - 167 : print out the projection parameters

168 - 197 : print out the numerical grid

The program contained on the next seven pages was used to produce Example 5 of this thesis.

SOURCE LANGUAGE STATEMENTS

```
    1                DIMENSION (X(190,I1),Y(190,I1),V(10),W(19),IMP(190,I1),RDM(190,I1),RDP(190,I1))-
    2                INTEGERS (T,U,V,W,BC)-
    3                FLOATING (LCR,LPR,NP,NC,IMP,NPB,LPRL,IFL)-
    4                PRECISION (2,X,Y,RAD,F,A,FLAT,E2,BCR,BPR,LCR,LPR,HC,NPB,DENOM,NP,NC,DSIN.,DCOS.,CHECK,DENO,BPRB,LPRL
                         ,XB,YB,DXB,DYB,AB,DXL,DYL,AL,DATAN.,XL,YL,DEN,IMP,DMM,DPM,DPE,DME,A1,B1,C1,D1,E1,RDM,RDP,CHECK,B
                         PR1,BPR2,BCR1,F1,DSQRT.)-
    5   START        RAD=3.1415926536/180.0-
    6                A=6378388.000-
    7                IFL=297.0-
    8                FLAT=1.0/IFL-
    9                HC=1126542.9-
   10                F=HC-
   11                BC=40-
   12                LC=90-
   13                DEGCH=0.00001-
   14                U=1-
   15                E2=2*FLAT-FLAT.P.2-
   16                I1=10-
   17                A1=1.0+(3.0/4.0)*E2+(45.0/64.0)*E2*E2+(175.0/256.0)*E2*E2*E2+(11025.0/16384.0)*E2*E2*E2*E2+(43659.0/6
                         5536.0)*E2*E2*E2*E2*E2-
   18                B1=(3.0/4.0)*E2+(15.0/16.0)*E2*E2+(525.0/512.0)*E2*E2*E2+(2205.0/2048.0)*E2*E2*E2*E2+(72765.0/65536.0
                         )*E2*E2*E2*E2*E2-
   19                C1=(15.0/64.0)*E2*E2+(105.0/256.0)*E2*E2*E2+(2205.0/4096.0)*E2*E2*E2*E2+(10395.0/16384.0)*E2*E2*E2*E2
                         *E2-
   20                D1=(35.0/512.0)*E2*E2*E2+(315.0/2048.0)*E2*E2*E2*E2+(31185.0/131072.0)*E2*E2*E2*E2*E2-
   21                E1=(315.0/16384.0)*E2*E2*E2*E2+(3465.0/65536.0)*E2*E2*E2*E2*E2-
   22                F1=(693.0/131072.0)*E2*E2*E2*E2*E2-
   23                BCR=BC*RAD-
   24                LCR=LC*RAD-
   25                I=9-
   26                W(I)=BC-
```

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```
               T=C-
               LP=C.0-
               BP=8C-
               BPR=BP*RAD-
               I3=1-
               I4=0-
               BPR1=89.9*RAD-
               BPR2=-89.9*RAD-
               BCR1=89.9*RAD-
      START1    GO THROUGH (LOOP),J=C,1,J.L.10-
               LPR=10.0*J*RAD*I3+180.0*RAD*I4-
               NP=A/(DSQRT.(1-E2*DSIN.(BPR).P.2))-
               NC=A/(DSQRT.(1-E2*DSIN.(BCR).P.2))-
                PROVIDED (U.L.1), TRANSFER TO (ORTHO)-
               DENOM=NC*(1-E2*DSIN.(BCR).P.2)-NP*(DCOS.(BCR)*DCOS.(BPR)*DCOS.(LPR)+(1-E2)*DSIN.(BCR)*DSIN.(BPR))*HC-
                TRANSFER TO (CALC)-
      ORTHO    DENOM=1.0-
      CALC     X(I,J)=(F*NP*DCOS.(BPR)*DSIN.(LPR))/DENOM-
      CONT     Y(I,J)=(F*(NC*E2*DSIN.(BCR)*DCOS.(BCR)+NP*((1-E2)*DCOS.(BCR)*DSIN.(BPR)-DSIN.(BCR)*DCOS.(BPR)*DCOS.(L
                    PR))))/DENOM-
                PROVIDED (BPR.G.89.0*RAD), TRANSFER TO (BPP90)-
                PROVIDED (BPR.L.-89.0*RAD), TRANSFER TO (BPM90)-
                PROVIDED (BCR.G.89.0*RAD), TRANSFER TO (BCP90)-
               CHECK=(A/(A+HC)-DSIN.(BCR)*DSIN.(BPR))/(DCOS.(BCR)*DCOS.(BPR))-
                TRANSFER TO (CHEK)-
      BPP90     PROVIDED (BCR.G.89.9*RAD), TRANSFER TO (BCP)-
               CHECK=(A/(A+HC)-DSIN.(BCR)*DSIN.(BPR))/(DCOS.(BCR)*DCOS.(BPR1))-
                TRANSFER TO (CHEK)-
      BCP      CHECK=(A/(A+HC)-DSIN.(BCR)*DSIN.(BPR))/(DCOS.(BCR1)*DCOS.(BPR1))-
                TRANSFER TO (CHEK)-
```

```
BPM9C    PROVIDED (BCR.G.89.9*RAD), TRANSFER TO (BCM)-
         CHECK=(A/(A+HC)-DSIN.(BCR)*DSIN.(BPR))/(DCOS.(BCR)*DCOS.(BPR2))-
         TRANSFER TO (CHEK)-
BCM      CHECK=(A/(A+HC)-DSIN.(BCR)*DSIN.(BPR))/(DCOS.(BCR1)*DCOS.(BPR2))-
         TRANSFER TO (CHEK)-
BCP9C    PROVIDED (BPR.G.89.9*RAD), TRANSFER TO (BCP)-
         PROVIDED (BPR.L.-89.9*RAD), TRANSFER TO (BCM)-
         CHECK=(A/(A+HC)-DSIN.(BCR)*DSIN.(BPR))/(DCOS.(BCR1)*DCOS.(BPR))-
CHEK     PROVIDED (DCOS.(LPR).GE.CHECK), TRANSFER TO (ANDIS)-
         X(I,J)=19191919-
         Y(I,J)=19191919-
         PROVIDED (J.L.5), TRANSFER TO (PAGE2)-
         TRANSFER TO (TGO)-
PAGE2    K=5-
         X(I,K)=19191919-
         TRANSFER TO (TGO)-
ANDIS    BPRB=BPR+(DEGCH*RAD*DCOS.(BPR))-
         PROVIDED (BPR.G.89.9*RAD.OR.BPR.L.-89.9*RAD), TRANSFER TO (OUT)-
         NPB=A/(DSQRT.(1-E2*DSIN.(BPRB).P.2))-
         PROVIDED (U.L.1), TRANSFER TO (ORTH)-
         DENO=NC*(1-E2*DSIN.(BCR).P.2)-NPB*(DCOS.(BCR)*DCOS.(BPRB)*DCOS.(LPR)+(1-E2)*DSIN.(BCR)*DSIN.(BPRB))+H
            C-
         TRANSFER TO (CAL)-
ORTH     DENO=1.0-
CAL      XB=(F*NPB*DCOS.(BPRB)*DSIN.(LPR))/DENO-
         YB=(F*(NC*E2*DSIN.(BCR)*DCOS.(BCR)+NPB*((1-E2)*DCOS.(BCR)*DSIN.(BPRB)-DSIN.(BCR)*DCOS.(BPRB)*DCOS.(LP
            R))))/DENO-
         LPRL=LPR+(DEGCH*RAD)-
         PROVIDED (U.L.1), TRANSFER TO (ORT)-
         DEN=NC*(1-E2*DSIN.(BCR).P.2)-NP*(DCOS.(BCR)*DCOS.(BPR)*DCOS.(LPRL)+(1-E2)*DSIN.(BCR)*DSIN.(BPR))+HC-
```

```
 84              TRANSFER TO (CA)-
 85   ORT       DEN=1.0-
 86   CA        XL=(F*NP*DCOS.(BPR)*DSIN.(LPRL))/DEN-
 87             YL=(F*(NC*E2*DSIN.(BCR)*DCOS.(BCR)+NP*((1-E2)*DCOS.(BCR)*DSIN.(BPR)-DSIN.(BCR)*DCOS.(BPR)*DCOS.(LPRL)
                     )))/DEN-
 88             DXB=.ABS.(XB-X(I,J))-
 89             DYB=.ABS.(YB-Y(I,J))-
 90             AB=DATAN.(DYB/DXB)-
 91   DAL       DXL=.ABS.(XL-X(I,J))-
 92             DYL=.ABS.(YL-Y(I,J))-
 93             AL=DATAN.(DYL/DXL)-
 94   CIMP1      PROVIDED (XB.G.X(I,J)), TRANSFER TO (CIMP2)-
 95              PROVIDED (YL.G.Y(I,J)), TRANSFER TO (CIMP3)-
 96             IMP(I,J)=.ABS.(180+(AL-AB)/RAD)-
 97              TRANSFER TO (CIMP)-
 98   CIMP3     IMP(I,J)=.ABS.(180-(AL+AB)/RAD)-
 99   CIMP      IMP(I,J)=.ABS.(IMP(I,J)-90.000000)-
100              TRANSFER TO (LENDIS)-
101   CIMP2      PROVIDED (YL.G.Y(I,J)), TRANSFER TO (CIMP4)-
102             IMP(I,J)=.ABS.((AB+AL)/RAD)-
103              TRANSFER TO (CIMP)-
104   CIMP4     IMP(I,J)=.ABS.((AB-AL)/RAD)-
105              TRANSFER TO (CIMP)-
106   LENDIS    DMM=(DXB.P.2+DYB.P.2).P.0.5-
107             DPM=(DXL.P.2+DYL.P.2).P.0.5-
108             DPE=.ABS.(NP*DCOS.(BPR)*DEGCH*RAD)-
109             DME=.ABS.(A*(1-E2)*(.ABS.(A1*(BPRB-BPR))-.ABS.((B1/2)*(DSIN.(2*BPRB)-DSIN.(2*BPR)))+.ABS.((C1/4)*(DSI
                     N.(4*BPRB)-DSIN.(4*BPR)))-.ABS.((D1/6)*(DSIN.(6*BPRB)-DSIN.(6*BPR)))+.ABS.((E1/8)*(DSIN.(8*BPRB)
                     -DSIN.(8*BPR)))-.ABS.((F1/10)*(DSIN.(10*BPRB)-DSIN.(10*BPR)))))-
110             RDM(I,J)=DMM/DME-
111             RDP(I,J)=DPM/DPE-
```

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```
              TRANSFER TO (LOOP)-
      OUT    IMP(I,J)=1.1111-
             RDM(I,J)=1.1111-
             RDP(I,J)=1.1111-
      LOOP    CONTINUE -
      TG0     PROVIDED (T.G.0), TRANSFER TO (TG1)-
             BPR=BPR+(10.0*RAD)-
              PROVIDED ((BPR/RAD).G.91.0), TRANSFER TO (TE1)-
             I=I+1-
              PROVIDED (I.G.18), TRANSFER TO (TE2)-
      WIEBP  W(I)=BPR/RAD-
              TRANSFER TO (START1)-
      TG1     PROVIDED (T.G.1), TRANSFER TO (TG2)-
              TRANSFER TO (BPM10)-
      TE1    T=1-
             I3=-1-
             I4=1-
             BPR=BPR-10.0*RAD-
      BPM10  BPR=BPR-10.0*RAD-
             I=I+1-
              PROVIDED (I.G.18), TRANSFER TO (TE2)-
              TRANSFER TO (WIEBP)-
      TG2     PROVIDED (T.G.2), TRANSFER TO (BPP10)-
              TRANSFER TO (BPM102)-
      TE2    T=2-
             I3=1-
             I4=0-
             I=9-
             BPR=BCR-
```

```
141  BPM102    BPR=BPR-(10.0*RAD)-
142             PROVIDED (BPR/RAD.L.-91.0), TRANSFER TO (TE3)-
143            I=I-1-
144             PROVIDED (I.L.0), TRANSFER TO (WRITE)-
145             TRANSFER TO (WIEBP)-
146  TE3       T=3-
147            I3=-1-
148            I4=1-
149            BPR=BPR+(10.0*RAD)-
150  BPP10     BPR=BPR+(10.0*RAD)-
151            I=I-1-
152             PROVIDED (I.L.0), TRANSFER TO (WRITE)-
153             TRANSFER TO (WIEBP)-
154  WRITE     I=0-
155            V(I)=LC-
156  IEIP1     I=I+1-
157             PROVIDED (I.G.9), TRANSFER TO (FINAL)-
158            V(I)=.ABS.(V(I-1)-10)-
159             TRANSFER TO (IEIP1)-
160  FINAL      WRITE NO HEADING ,FORM90-
161 F FORM90   (1H1//////////33X,27HPERSPECTIVE MAP PROJECTIONS/)    -
162             WRITE OUTPUT ,FORM91-
163 F FORM91   (32X,30HEXAMPLE 5 - GENERAL PROJECTION///////)  -
164             WRITE OUTPUT ,FORM92,(BC,LC,HC,F)-
165 F FORM92   (22X,23HPROJECTION PARAMETERS -//33X,19HCENTRAL LATITUDE = ,I3,8H DEGREES//33X,20HCENTRAL LONGITUDE =
                 ,I3,8H DEGREES//33X,30HHEIGHT OF PROJECTION CENTER = ,DF13.3,7H METERS//33X,30HFOCAL LENGTH (SCA
                 LE FACTOR) = ,DF13.3,7H METERS/)     -
166             WRITE OUTPUT ,FORM94,(A,IFL)-
167 F FORM94   (33X,32HELLIPSOID MAJOR SEMI DIAMETER = ,DF13.3,7H METERS//33X,25HELLIPSOID FLATTENING = 1/,F5.1)
                 -
168             WRITE NO HEADING ,FORM8,((V(I),I=0,1,I.L.5))-
```

```
  F FORM8   - (1H1,11X,14HLONGITUDE.....,I3,10X,I3,10X,I3,10X,I3,10X,I3//)-
                 I=15-
    START2       J=0-
                  PROVIDED (X(I,J).E.19191919), TRANSFER TO (SUB1)-
                  WRITE OUTPUT ,FORM6,(W(I),(X(I,J),J=0,1,X(I,J).NE.19191919.AND.J.L.5))-
  F FORM6       (12X,5HLAT. ,I3/12X7H   X   ,5DF13.2)      -
                  WRITE OUTPUT ,FORM7,((Y(I,J),J=0,1,X(I,J).NE.19191919.AND.J.L.5))-
  F FORM7       (12X,7H   Y   ,5DF13.2) -
                  WRITE OUTPUT ,FORM12,((RDM(I,J),J=0,1,X(I,J).NE.19191919.AND.J.L.5))-
  F FORM12      (12X,7HMER.   ,5DF13.4) -
                  WRITE OUTPUT ,FORM22,((RDP(I,J),J=0,1,X(I,J).NE.19191919.AND.J.L.5))-
  F FORM22      (12X,7HPAR.   ,5DF13.4) -
                  WRITE OUTPUT ,FORM23,((IMP(I,J),J=0,1,X(I,J).NE.19191919.AND.J.L.5))-
  F FORM23      (12X,7HANG.   ,5DF13.4) -
    SUB1         I=I-1-
                  PROVIDED (I.GE.0), TRANSFER TO (START2)-
                  WRITE NO HEADING ,FORM8,((V(I),I=5,1,I.L.10))-
                 I=18-
    START4       J=5-
                  PROVIDED (X(I,J).E.19191919), TRANSFER TO (SUB2)-
                  WRITE OUTPUT ,FORM6,(W(I),(X(I,J),J=5,1,X(I,J).NE.19191919.AND.J.L.10))-
                  WRITE OUTPUT ,FORM7,((Y(I,J),J=5,1,X(I,J).NE.19191919.AND.J.L.10))-
                  WRITE OUTPUT ,FORM12,((RDM(I,J),J=5,1,X(I,J).NE.19191919.AND.J.L.10))-
                  WRITE OUTPUT ,FORM22,((RDP(I,J),J=5,1,X(I,J).NE.19191919.AND.J.L.10))-
                  WRITE OUTPUT ,FORM23,((IMP(I,J),J=5,1,X(I,J).NE.19191919.AND.J.L.10))-
    SUB2         I=I-1-
                  PROVIDED (I.GE.0), TRANSFER TO (START4)-
    END           CALL SUBROUTINE ()=ENDJOB.()-
                  END PROGRAM (START)-
```

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